Geometric Control of Quantum Spin Systems

Christopher G. Moseley
Department of Mathematical Sciences
United States Military Academy, West Point, NY

Howard E. Brandt
Sensors and Electron Devices Directorate
U. S. Army Research Laboratory, Adelphi, MD

November 5, 2003

Quantum Computers:

- use ‘qubits’ instead of classical bits
- qubits can be built from quantum spin systems
- actual implementation is difficult
- potential rewards are great

Shor (1997): algorithm for prime factorization on a quantum computer

- much more efficient than any known classical algorithm
- many encryption schemes rely on difficulty of factoring large numbers

Quantum Spin Systems

- state vector evolves in accordance with quantum mechanics
- desired evolution of a system can be corrupted by quantum decoherence
- only operational physical implementation to date: NMR spin system
- strategy: find optimal paths in state space to minimize decoherence

Control of NMR spin systems

- Khaneja, Brockett and Glaser (2001): control problem reduces to finding sub-Riemannian geodesics on state space
- we propose a method for extending their solution to systems with $n > 3$ qubits

Quantum Mechanics Background

The state vector $|\psi(t)>$ of a quantum system with n qubits is given by

$$|\psi(t)> = U(t)|\psi(0)> \tag{1}$$

where $|\psi(0)>$ is the initial state and $U(t) \in SU(2^n)$ evolves according to the time dependent Schrödinger equation

$$\dot{U}(t) = -\frac{i}{h} H(t) U(t). \tag{2}$$

H is called the *Hamiltonian* of the system.

The Hamiltonian of an NMR system can be decomposed as

$$H = H_d + \sum_{j=1}^{m} u_j H_j \tag{3}$$

where H_d is the *drift* Hamiltonian (internal couplings), the u_j are *controls* and the H_j are the *rf* or *control* Hamiltonians.

The control Hamiltonians can be chosen so that the $\{iH_j\}$ generate the Lie algebra of a closed Lie subgroup $K \subset SU(2^n)$.

$$H = H_d + \sum_{j=1}^{m} u_j H_j$$

The controls u_j can be made so large that the time needed to transition between two elements U_A, U_B in the same coset

$$KU_A = \{kU_A : k \in K\}$$

is negligible; i.e. too small for the system to evolve substantially under the drift Hamiltonian H_d.

The optimal control problem is therefore:

find the shortest path between cosets in $SU(2^n)/K$.

Khaneja et al showed (in the two-qubit case) that this is equivalent to finding *sub-Riemannian geodesics* on $SU(4)/(SU(2) \otimes SU(2))$. The methods they used for finding these geodesics took advantage of the fact that $SU(4)/(SU(2) \otimes SU(2))$ is a symmetric space, but this is not true for the state space in the general case.

Thus the problem would benefit from application of a method that is more generally applicable.

Brief synopsis of sub-Riemannian geometry

In NMR systems only a sub-bundle D of the tangent bundle TM of the state space $M = SU(2^n)/(SU(2) \otimes \ldots \otimes SU(2))$ is accessible.

A smooth inner product $\langle\, , \,\rangle$ on D is called a *sub-Riemannian metric.* Admissible paths $\gamma : [a, b] \to M$ that minimize the length functional

$$L(\gamma) = \int_a^b \langle \dot\gamma(t), \dot\gamma(t) \rangle^{1/2}\, dt \tag{4}$$

are *sub-Riemannian geodesics.*

Griffiths formalism for constrained optimization

Let $X = D \times \mathbb{R}$ and let ϕ be the 1-form

$$\phi = \frac{1}{2}((p^1)^2 + (p^2)^2)\, dt \tag{5}$$

on X. Let $Z \subset T^*X$ be the submanifold defined by

$$Z = \bigcup_{x \in X} Z_x$$

where $Z_x = \{\phi(x) + I_x \subset T^*_x X\}$, and I is the defining coframing of D lifted to X.

Griffiths formalism, cont'd

The constrained variational problem on the state space M is thus lifted to an unconstrained variational problem on Z.

The integral curves of the Cartan system of the canonical symplectic 2-form on Z project to regular sub-Riemannian geodesics on the state space M.

Griffiths formalism, cont'd

This method was used by the first author in his Ph.D. thesis to find sub-Riemannian geodesics on Engel 4-manifolds. In particular, he found explicit equations for Engel systems on the Lie groups $SO(3) \times S^1$, $SEuc(2) \times S^1$, and $SO(2,1) \times S^1$.

- **Advantages:**
  - applicable to quantum control problems with any finite number of spins
  - first author has applied it in the 4-dimensional case already
- **Disadvantages:** number of differential equations becomes rapidly larger as the number of spins increases

Summary:

- it is possible to extend Khaneja et al's results, with the help of differential geometric methods as outlined above.
- it will take a good deal of work
- the possible rewards are worthwhile, since control of substantial numbers of spin-based qubits is required for useful quantum information processing