Quasi-Analytical Shock Response Spectra, for Proportionally Damped Systems, with Remote Shock Loading

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Recently, exact expressions were presented for the time-domain response of an MSD system to a seismic input comprising exponentially decaying sinusoids. Such an input could result from a remote shock consisting of an ideal impulse, as filtered through the eigensystem comprising the linear model of a structure. In this paper the analytical time response, of the (hypothetical) MSD system used for SRS determination, is found for a remote shock input of three forms: an ideal impulse, a rectangular pulse, and a saw-tooth pulse. These responses permit exact SRS determination without necessitating numerical evaluation of a convolution integral.

INTRODUCTION

Modern warfare calls for many military systems to be capable of sustained operation under extreme environmental conditions. Designers of military equipment must typically harden their hardware to maintain an acceptable degree of functionality when exposed to mechanical shock, from such sources as blast-waves, collisions, and projectile impacts. Frequently the military and its vendors define a design shock environment in terms of the maximum kinematic response, over time, that it will produce in a hypothetical, single-degree-of-freedom (SDOF) mass-spring-damper (MSD) system attached at the point of a kinematic disturbance. (See Fig. 1.) The disturbance is typically a

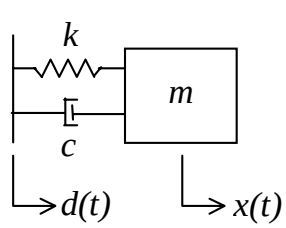

Figure 1. Hypothetical SDOF MSD system, for SRS determination

direct impulsive displacement or acceleration, not necessarily known; and the response, the SDOF-MSD mass' displacement (absolute or relative), pseudo-velocity (relative displacement multiplied by the undamped natural frequency of the SDOF MSD system), or absolute acceleration. A quantified representation (e.g., by plot, equation,

or table) of the maximum selected response, as a function of frequency, is known as a shock response spectrum (SRS).

For a SDOF MSD system disturbed by a direct input $d(t)$ (at *D*, Fig. 2), comprising a linear combination of exponentially decaying sinusoids all beginning at time zero, the displacement response $x(t)$ has a relatively simple analytical form [1]. In particular, the displacement response comprises a linear combination of exponentially decaying, phase-shifted sinusoids, at the forcing frequencies and at the damped natural frequency for the hypothetical SDOF MSD system. Like the direct-input sinusoids, the response sinusoids all begin at time zero. The input disturbance can be described as "local" (since it is applied locally, at *D*), and can be regarded as a filtered, or induced, input due to a remote shock (e.g., at *C*, Fig. 2). As will be shown below, for a linear system with constant system matrices (mass, stiffness, and damping) the local input $d(t)$ will have the indicated form, provided the remote shock input is an ideal impulse (i.e., a Dirac-delta function). It can be shown as well—but is outside the scope of this paper—that proportional (Rayleigh) damping is not required for this result.

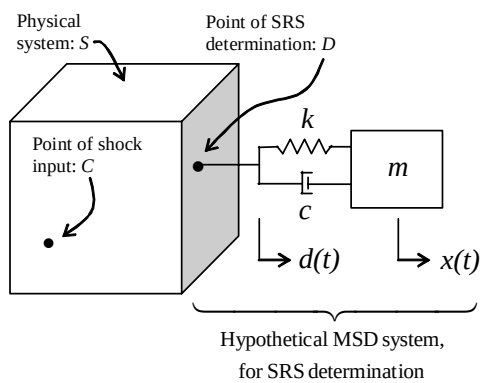

Figure 2. Shock-loaded system *S*, with (hypothetical) attached MSD system for SRS determination

Since a physical shock input occurs over finite time, it is reasonable to seek SDOF-MSD system responses to remote inputs other than the Dirac delta. The present work will treat three types of remote inputs, in ascending order of difficulty: the ideal impulse (which is mathematically the simplest), the rectangular pulse (the simplest input occurring in finite time), and the saw-tooth pulse (the most physically realistic). For each remote input analytical expressions will be found for local (induced, filtered) input $d(t)$, displacement response $x(t)$, and relative displacement $\delta(t)$.

PROBLEM STATEMENT

Consider a linear, proportionally damped physical system (represented pictorially by *S* in Fig. 2), with *n* degrees of freedom (*n*-DOFs), having mass, damping, and stiffness matrices represented, respectively, by *M*, *C*, and *K*. For forcing vector $\underline{f}$ and disturbance input matrix *E*, the 2nd-order matrix differential equation of motion can be expressed by

$$M\underline{\ddot{x}} + C\dot{x} + Kx = E\underline{f}\,, \tag{1}$$

where $\underline{x}$, $\underline{\dot{x}}$, and $\underline{\ddot{x}}$ are vectors representing the generalized physical coordinates, velocities, and accelerations, respectively. Assume initial conditions $\underline{x}(0)$ and $\underline{\dot{x}}(0)$. Assume also the equations to be arranged such that the jth coordinate x_j is the scalar response d of *S*, at point *D* (Fig. 2), to input scalar disturbance f_k applied at point *C*, with all other elements of input vector $\underline{f}$ set at zero. The response of the attached SDOF MSD system to this input is x (Fig. 2). Assume further that mass *m* is infinitesimally small, so that the motion of *m* will not affect that of *D*. (I.e., there will be no output feedback, or output impedance, to *S*).

The objectives of this paper are to determine analytical expressions for the absolute displacements $d(t)$ and $x(t)$, and for relative displacement

$$\delta = x - d \ , \tag{2}$$

for shock input $f_k(t)$ described variously by an ideal impulse, a rectangular pulse, and a sawtooth pulse. Note that the shock input is "remote," in that points *C* and *D* are not coincident. The displacement *d* is both a response of *S* at *D* and a local, "induced" input to the attached SDOF MSD system; it is a "filtered" input, consisting of the shock input f_k filtered through the eigensystem representing *S*.

GENERAL FORCING

Modal Response of Basic System

From (1) the homogeneous differential equation of motion for the undamped system *S* is

$$M\,\underline{\ddot{x}} + Kx = \underline{0} \ . \tag{3}$$

Let the mass-normalized modeshape matrix for *S* be

$$\tilde{U} = \left[\underline{\tilde{m}}_1, \cdots, \underline{\tilde{m}}_n\right], \tag{4}$$

where vector $\underline{\tilde{m}}_i$ represents the ith individual mass-normalized modeshape. Using the modal coordinate vector $\underline{\eta}$ as defined by the relationship [2]

$$\underline{x} = \tilde{U}\,\underline{\eta} \ , \tag{5}$$

one can decouple (1) into the following *n* SDOF modal equations of motion:

$$\ddot{\eta}_i + 2\varsigma_i\omega_{ni}\dot{\eta}_i + \omega_{ni}^2\eta_i = \tilde{f}_i \ , \tag{6}$$

where η_i, ω_{ni}, ς_i, and $\tilde{f}_i$ are, respectively, the modal coordinate, modal undamped natural frequency, modal damping ratio, and modal force for the ith mode. The ith modal damped natural frequency can be represented correspondingly by ω_{di}.

Using Laplace transforms, and assuming all modes to be underdamped, the solution to (6) is

$$\eta_i(t) = e^{-\varsigma_i\omega_{ni}t}\left(A_i\cos\omega_{di}t + B_i\sin\omega_{di}t\right)u_{-1}(t) + \frac{1}{\omega_{di}}\left(e^{-\varsigma_i\omega_{ni}t}\sin\omega_{di}t\right) * \tilde{f}_i\,u_{-1}(t), \tag{7}$$

where

$$A_i = \eta_i\left(0^+\right), \tag{8}$$

$$B_i = \frac{\varsigma_i\omega_{ni}}{\omega_{di}}\eta_i\left(0^+\right) + \frac{1}{\omega_{di}}\dot{\eta}_i\left(0^+\right), \tag{9}$$

and

$$\tilde{f}_i = \underline{\tilde{m}}_i^T E\,\underline{f} \ . \tag{10}$$

The asterisk in (7) represents convolution in the time domain, and $u_{-1}(t)$ represents the unit step applied an instant after time zero, at $t = 0^+$. For forcing vector $\underline{f}$ consisting of all zero elements except for a kth element of form $f_k u_{-1}(t)$, Equation (7) can be rewritten as

$$\eta_i(t) = e^{-\varsigma_i \omega_{ni} t}\left(A_i \cos\omega_{di} t + B_i \sin\omega_{di} t\right) u_{-1}(t) + \frac{\mu_{ik}}{\omega_{di}}\left(e^{-\varsigma_i \omega_{ni} t} \sin\omega_{di} t\right) * f_k u_{-1}(t), \tag{11}$$

where

$$\mu_{ik} = \left(\underline{\tilde{m}}_i^T E\right)_k \tag{12}$$

designates the kth element of $\underline{\tilde{m}}_i^T E$.

Physical Response of Basic System

The induced displacement $d(t)$ is found by applying (4) and (7) to (5), and taking the jth element of the resulting coordinate vector:

$$d(t) = (\underline{x})_j = \left(\sum_{i=1}^{n} \eta_i \underline{\tilde{m}}_i\right)_j, \tag{13}$$

where $\eta_i(t)$ is given by (7). Accordingly,

$$d(t) = \sum_{i=1}^{n} \eta_i \tilde{m}_{ij}, \tag{14}$$

where $\tilde{m}_{ij}$ is the jth element of the ith modeshape $\underline{\tilde{m}}_i$.

Displacement of Attached Mass

The attached-mass response $x(t)$ to induced input $d(t)$ has been shown [1] (using Laplace transforms) to be

$$\begin{aligned} x(t) = {} & e^{-\varsigma\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \\ & + C\left[e^{-\varsigma\omega_n t}\sin\left(\omega_d t + \phi\right)\right] * d(t), \end{aligned} \tag{15}$$

where

$$\omega_d = \omega_n\sqrt{1-\varsigma^2}, \tag{16}$$

$$A = x(0+), \tag{17}$$

$$B = \frac{\varsigma\omega_n}{\omega_d} x(0+) + \frac{1}{\omega_d}\dot{x}(0+) - \frac{2\varsigma\omega_n}{\omega_d} d(0+), \tag{18}$$

$$C = \frac{\omega_n^2}{\omega_d}, \quad (19)$$

and

$$\phi = \tan^{-1}\left(\frac{2\varsigma\sqrt{1-\varsigma^2}}{1-2\varsigma^2}\right). \tag{20}$$

Note that, whereas parameters ς_i, ω_{ni}, and ω_{di} pertain to the physical system S (Figure 2), parameters ς, ω_n, and ω_d describe the attached (conceptual) SDOF system. Upon applying (14) and (11) to (15), mass *m* is found to have the following response to general remote shock input $f_k u_{-1}(t)$:

$$x(t) = e^{-\varsigma\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + C\left[e^{-\varsigma\omega_n t}\sin\left(\omega_d t + \phi\right)\right] * \sum_{i=1}^{n} \eta_i \tilde{m}_{ij}, \tag{21}$$

where η_i is given by (11) and (12).

Remarks

One can note from (7), (14), and (15) that determining $x(t)$ requires performing two consecutive convolutions. Due to the algebraic complexity involved, for SRS evaluation these convolutions are normally accomplished numerically. However, with certain simple shock inputs the convolution integrals can be evaluated algebraically, to provide full algebraic solutions for $x(t)$. From (11) and (14) it can be determined that for a Dirac-delta shock input f_k, the induced displacement $d(t)$ will be a linear combination of damped sinusoids, the local input treated previously in [1]. This simplest case will be addressed first below. More complicated (and realistic) shock inputs will be addressed in the subsequent two sections.

SOLUTION FOR AN IDEAL IMPULSE

Remote (Initial, Unfiltered) Input $f_k(t)$

Consider the case of an ideal impulse of strength γ at point *C* (Fig. 2):

$$f_k(t) = \gamma\, u_0(t). \tag{22}$$

Local (Induced, Filtered) Input $d(t)$

For this remote input, from (11) and (14) the local (filtered, induced) input reduces to

$$d(t) = \left[\sum_{i=1}^{n} D_{i1} e^{-\varsigma_i \omega_{ni} t} \cos\omega_{di} t + \sum_{i=1}^{n} D_{i2} e^{-\varsigma_i \omega_{ni} t} \sin\omega_{di} t\right] u_{-1}(t), \tag{23}$$

where

$$D_{i1} = \tilde{m}_{ij} A_i, \tag{24}$$

and

$$D_{i2} = \tilde{m}_{ij}\left(B_i + \mu_{ik}\gamma/\omega_{di}\right). \tag{25}$$

Displacement Response $x(t)$

Since the induced input is a linear combination of exponentially decaying sinusoids, the displacement response can be determined using the results of Reference [1], to obtain the following:

$$\begin{aligned} x(t) = \Big\langle e^{-\varsigma\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) &+ C\sum_{i=1}^{n} D_{i1}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\cos\left(\omega_d t + \phi + \theta_{1i}\right) - \delta_{2i}\cos\left(\omega_d t + \phi + \theta_{2i}\right)\right] \\ &\quad + e^{-\varsigma_i\omega_{ni} t}\left[\delta_{1i}\cos\left(\omega_{di} t + \phi + \theta_{1i}\right) - \delta_{2i}\cos\left(\omega_{di} t - \phi + \theta_{3i}\right)\right]\Big\} \\ &+ C\sum_{i=1}^{n} D_{i2}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\sin\left(\omega_d t + \phi + \theta_{1i}\right) + \delta_{2i}\sin\left(\omega_d t + \phi + \theta_{2i}\right)\right] \\ &\quad + e^{-\varsigma_i\omega_{ni} t}\left[\delta_{1i}\sin\left(\omega_{di} t + \phi + \theta_{1i}\right) - \delta_{2i}\sin\left(\omega_{di} t - \phi + \theta_{3i}\right)\right]\Big\}\Big\rangle u_{-1}(t), \end{aligned} \tag{26}$$

where

$$\delta_{1i} = \frac{1}{2\sqrt{\mu_i^2 + \nu_i^2}}, \tag{27}$$

$$\delta_{2i} = \frac{1}{2\sqrt{\mu_i^2 + \rho_i^2}}, \tag{28}$$

$$\mu_i = \varsigma_i\omega_{ni} - \varsigma\omega_n, \tag{29}$$

$$\nu_i = \omega_d - \omega_{di}, \tag{30}$$

$$\rho_i = \omega_d + \omega_{di}, \tag{31}$$

$$\theta_{1i} = \tan^{-1}\left(\mu_i/\nu_i\right), \tag{32}$$

$$\theta_{2i} = \tan^{-1}\left(\mu_i / \rho_i\right), \tag{33}$$

$$\theta_{3i} = \tan^{-1}\left(\frac{\mu_i}{-\rho_i}\right), \tag{34}$$

and the other variables are as defined previously.

Relative Displacement $\delta(t)$

Subtracting (23) from (26), one obtains an expression for the relative displacement:

$$\begin{aligned}
\delta(t) = x(t) - d(t) \; &= \Big\langle e^{-\varsigma\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \\
&+ C\sum_{i=1}^{n} D_{i1}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\cos\left(\omega_d t + \phi + \theta_{1i}\right) - \delta_{2i}\cos\left(\omega_d t + \phi + \theta_{2i}\right)\right] \\
&\qquad + e^{-\varsigma_i\omega_{ni} t}\left[\delta_{1i}\cos\left(\omega_{di} t + \phi + \theta_{1i}\right) - \delta_{2i}\cos\left(\omega_{di} t - \phi + \theta_{3i}\right) \; - (1/C)\cos\omega_{di} t\right]\Big\} \\
&+ C\sum_{i=1}^{n} D_{i2}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\sin\left(\omega_d t + \phi + \theta_{1i}\right) + \delta_{2i}\sin\left(\omega_d t + \phi + \theta_{2i}\right)\right] \\
&\qquad + e^{-\varsigma_i\omega_{ni} t}\left[\delta_{1i}\sin\left(\omega_{di} t + \phi + \theta_{1i}\right) - \delta_{2i}\sin\left(\omega_{di} t - \phi + \theta_{3i}\right) \; - (1/C)\sin\omega_{di} t\right]\Big\}\Big\rangle u_{-1}(t)
\end{aligned} \tag{35}$$

SOLUTION FOR A RECTANGULAR PULSE

Remote Input $f_k(t)$

Consider next the case of a shock input idealized as a rectangular pulse of height γ and duration t_1 at point *C*:

$$f_k(t) = \gamma\left[u_{-1}(t) - u_{-1}(t - t_1)\right]. \tag{36}$$

Local Input $d(t)$

Substitution of (36) into (11), yields the following:

$$\eta_i(t) = e^{-\varsigma_i\omega_{ni} t}\left(A_i\cos\omega_{di} t + B_i\sin\omega_{di} t\right)u_{-1}(t) + \left(\mu_{ik}/\omega_{di}\right)\left(e^{-\varsigma_i\omega_{ni} t}\sin\omega_{di} t\right) * \gamma\left[u_{-1}(t) - u_{-1}(t - t_1)\right] . \tag{37}$$

Let the convolution term be represented by

$$\eta_{i,\text{ forced}} = \left(\mu_{ik}/\omega_{di}\right)\left(e^{-\varsigma_i\omega_{ni} t}\sin\omega_{di} t\right) * \gamma\left[u_{-1}(t) - u_{-1}(t - t_1)\right] , \tag{38}$$

for which the Laplace transform ([3], pp. 655-656, nos. 5 and 24a) is

$$\mathrm{H}_{i,\text{ forced}}(s) = \frac{\mu_{ik}\gamma}{s\left(s^2 + 2\varsigma_i\omega_{ni} s + \omega_{ni}^2\right)}\left(1 - e^{-t_1 s}\right). \tag{39}$$

Taking the inverse Laplace transform ([3], p. 657, no. 27a) leads to the following time response:

$$\eta_{i,\text{ forced}} = \left[E_i + F_i e^{-\varsigma_i\omega_{ni} t}\sin\left(\omega_{di} t + \phi_i\right)\right]u_{-1}(t) - \left\{E_i + F_i e^{-\varsigma_i\omega_{ni}(t - t_1)}\sin\left[\omega_{di}(t - t_1) + \phi_i\right]\right\}u_{-1}(t - t_1), \tag{40}$$

where $$E_i = \mu_{ik}\gamma \big/ \omega_{ni}^2 \,, \tag{41}$$

$$F_i = \frac{-\mu_{ik}\gamma}{\omega_{ni}\omega_{di}} \,, \tag{42}$$

and $$\phi_i = \cos^{-1}\varsigma_i \;\left(i = 1, \cdots, \nu\right). \tag{43}$$

Accordingly, the complete response for the ith modal coordinate of S is

$$\eta_i(t) = e^{-\varsigma_i\omega_{ni}t}\left(A_i\cos\omega_{di}\,t + B_i\sin\omega_{di}\,t\right)u_{-1}(t) + \left[E_i + F_i e^{-\varsigma_i\omega_{ni}t}\sin\left(\omega_{di}t + \phi_i\right)\right]u_{-1}(t) - \left[E_i + F_i e^{-\varsigma_i\omega_{ni}(t-t_1)}\sin\left(\omega_{di}(t-t_1) + \phi_i\right)\right]u_{-1}(t-t_1)\,, \tag{44}$$

Substituting (44) into (14), the filtered input is found to be

$$\begin{aligned} d(t) = &\left[\sum_{i=1}^{n} D_{i1}e^{-\varsigma_i\omega_{ni}t}\cos\omega_{di}t + \sum_{i=1}^{\nu} D_{i2}e^{-\varsigma_i\omega_{ni}t}\sin\omega_{di}t\right]u_{-1}(t) + \left[\sum_{i=1}^{n} D_{i3}e^{-\varsigma_i\omega_{ni}t}\sin\left(\omega_{di}t + \phi_i\right)\right]u_{-1}(t) \\ &- \left[\sum_{i=1}^{n} D_{i3}e^{-\varsigma_i\omega_{ni}(t-t_1)}\sin\left(\omega_{di}(t-t_1) + \phi_i\right)\right]u_{-1}(t-t_1) + \left[\sum_{i=1}^{n} D_{i4}\right]\left[u_{-1}(t) - u_{-1}(t-t_1)\right], \end{aligned} \tag{45}$$

where $$D_{i1} = \tilde{m}_{ij}A_i \;\text{ (as before)}, \tag{46}$$

$$D_{i2} = \tilde{m}_{ij}B_i \;\text{ [contrast (25)]}, \tag{47}$$

$$D_{i3} = \tilde{m}_{ij}F_i \,, \tag{48}$$

and $$D_{i4} = \tilde{m}_{ij}E_i \,. \tag{49}$$

The remaining variables are as previously defined.

Displacement Response $x(t)$

Note that for this second description of shock load, the induced input $d(t)$ is a linear combination of two types of terms: exponentially decaying sinusoids (the first three summations of (45)), and a rectangular pulse term (the fourth summation). For the sinusoidal terms, the displacement response can be determined using the results of Reference [1]. For the classical pulse term, algebraic integration of the associated convolution can be accomplished via Laplace transforms. Using (15), the forced response due to the rectangular pulse is

$$x_{\text{pulse}}(t) = C\left[e^{-\varsigma\omega_n t}\sin\left(\omega_d t + \phi\right)\right] * \sum_{i=1}^{n} D_{i4}\left[u_{-1}(t) - u_{-1}(t-t_1)\right]. \tag{50}$$

Taking the Laplace transform ([3], p. 657, no. 26),

$$X_{\text{pulse}}(s) = C\frac{2\varsigma\omega_n s + \omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}\cdot\sum_{i=1}^{n}\frac{D_{i4}}{s}\left(1 - e^{-t_1 s}\right). \tag{51}$$

Rearranging, $$X_{\text{pulse}}(s) = \left(C\sum_{i=1}^{n} D_{i4}\right)\frac{2\varsigma\omega_n s + \omega_n^2}{s\left(s^2 + 2\varsigma\omega_n s + \omega_n^2\right)}\cdot\left(1 - e^{-t_1 s}\right). \tag{52}$$

Using Laplace-transform tables ([3], p. 657, no. 28) the corresponding time-domain contribution to the forced response can be determined:

$$x_{\text{pulse}}(t) = \left(C\sum_{i=1}^{n} D_{i4}\right)\left\{\left[1 + \left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n t}\sin\left(\omega_d t + \phi\right)\right]u_{-1}(t) - \left[1 + \left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n(t-t_1)}\sin\left(\omega_d(t-t_1) + \phi\right)\right]\right\}u_{-1}(t-t_1). \tag{53}$$

The full displacement response, then, is as follows:

$$
\begin{aligned}
x(t) = \Big\langle & e^{-\varsigma\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \\
& + C\sum_{i=1}^{n} D_{i1}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\cos\left(\omega_d t+\phi+\theta_{1i}\right) - \delta_{2i}\cos\left(\omega_d t+\phi+\theta_{2i}\right)\right] \\
& \qquad + e^{-\varsigma_i\omega_{ni} t}\left[\delta_{1i}\cos\left(\omega_{di} t+\phi+\theta_{1i}\right) - \delta_{2i}\cos\left(\omega_{di} t-\phi+\theta_{3i}\right)\right]\Big\} \\
& + C\sum_{i=1}^{n} D_{i2}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\sin\left(\omega_d t+\phi+\theta_{1i}\right) + \delta_{2i}\sin\left(\omega_d t+\phi+\theta_{2i}\right)\right] \\
& \qquad + e^{-\varsigma_i\omega_{ni} t}\left[\delta_{1i}\sin\left(\omega_{di} t+\phi+\theta_{1i}\right) - \delta_{2i}\sin\left(\omega_{di} t-\phi+\theta_{3i}\right)\right]\Big\}\Big\rangle u_{-1}(t) \\
& + C\sum_{i=1}^{n} D_{i3}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\sin\left(\omega_d t+\phi+\phi_i+\theta_{1i}\right) + \delta_{2i}\sin\left(\omega_d t+\phi-\phi_i+\theta_{2i}\right)\right] \\
& \qquad + e^{-\varsigma_i\omega_{ni} t}\left[\delta_{1i}\sin\left(\omega_{di} t+\phi_i+\phi+\theta_{1i}\right) - \delta_{2i}\sin\left(\omega_{di} t+\phi_i-\phi+\theta_{3i}\right)\right]\Big\}\Big\rangle u_{-1}(t) \\
& - \Bigg\langle C\sum_{i=1}^{n} D_{i3}\Big\{e^{-\varsigma\omega_n (t-t_1)}\left[-\delta_{1i}\sin\left(\omega_d (t-t_1)+\phi+\phi_i+\theta_{1i}\right) + \delta_{2i}\sin\left(\omega_d (t-t_1)+\phi-\phi_i+\theta_{2i}\right)\right] \\
& \qquad + e^{-\varsigma_i\omega_{ni} (t-t_1)}\left[\delta_{1i}\sin\left(\omega_{di} (t-t_1)+\phi_i+\phi+\theta_{1i}\right) - \delta_{2i}\sin\left(\omega_{di}(t-t_1)+\phi_i-\phi+\theta_{3i}\right)\right]\Big\}\Bigg\rangle u_{-1}(t-t_1) \\
& + \left(C\sum_{i=1}^{n} D_{i4}\right)\Big\{\left[1+\left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n t}\sin\left(\omega_d t+\phi\right)\right]u_{-1}(t) - \left[1+\left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n (t-t_1)}\sin\left(\omega_d (t-t_1)+\phi\right)\right]\Big\}u_{-1}(t-t_1). \qquad (54)
\end{aligned}
$$

Relative Displacement $\delta(t)$

Subtracting (45) from (54), one finally obtains an expression for the relative displacement:

$$
\begin{aligned}
\delta(t) = x(t) - d(t) \; = \Big\langle & e^{-\varsigma\omega_n t}\left(A\cos\omega_d t + B\sin\omega_d t\right) \\
& + C\sum_{i=1}^{n} D_{i1}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\cos\left(\omega_d t+\phi+\theta_{1i}\right) - \delta_{2i}\cos\left(\omega_d t+\phi+\theta_{2i}\right)\right] \\
& \qquad + e^{-\varsigma_i\omega_{ni} t}\left[\delta_{1i}\cos\left(\omega_{di} t+\phi+\theta_{1i}\right) - \delta_{2i}\cos\left(\omega_{di} t-\phi+\theta_{3i}\right) - \left(1/C\right)\cos\omega_{di} t\right]\Big\} \\
& + C\sum_{i=1}^{n} D_{i2}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\sin\left(\omega_d t+\phi+\theta_{1i}\right) + \delta_{2i}\sin\left(\omega_d t+\phi+\theta_{2i}\right)\right] \\
& \qquad + e^{-\varsigma_i\omega_{ni} t}\left[\delta_{1i}\sin\left(\omega_{di} t+\phi+\theta_{1i}\right) - \delta_{2i}\sin\left(\omega_{di} t-\phi+\theta_{3i}\right) - \left(1/C\right)\sin\omega_{di} t\right]\Big\} \\
& + C\sum_{i=1}^{n} D_{i3}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\sin\left(\omega_d t+\phi+\phi_i+\theta_{1i}\right) + \delta_{2i}\sin\left(\omega_d t+\phi-\phi_i+\theta_{2i}\right)\right] \\
& \qquad + e^{-\varsigma_i\omega_{ni} t}\left[\delta_{1i}\sin\left(\omega_{di} t+\phi_i+\phi+\theta_{1i}\right) - \delta_{2i}\sin\left(\omega_{di} t+\phi_i-\phi+\theta_{3i}\right) - \left(1/C\right)\sin\left(\omega_{di} t+\phi_i\right)\right]\Big\}\Big\rangle u_{-1}(t) \\
& - \Bigg\langle C\sum_{i=1}^{n} D_{i3}\Big\{e^{-\varsigma\omega_n (t-t_1)}\left[-\delta_{1i}\sin\left(\omega_d (t-t_1)+\phi+\phi_i+\theta_{1i}\right) + \delta_{2i}\sin\left(\omega_d (t-t_1)+\phi-\phi_i+\theta_{2i}\right)\right] \\
& \qquad + e^{-\varsigma_i\omega_{ni} (t-t_1)}\left[\delta_{1i}\sin\left(\omega_{di} (t-t_1)+\phi_i+\phi+\theta_{1i}\right) - \delta_{2i}\sin\left(\omega_{di}(t-t_1)+\phi_i-\phi+\theta_{3i}\right)\right. \\
& \qquad\qquad \left. - \left(1/C\right)\sin\left(\omega_{di} (t-t_1)+\phi_i\right)\right]\Big\}\Bigg\rangle u_{-1}(t-t_1) \\
& + \left(C\sum_{i=1}^{n} D_{i4}\right)\Big\{\left[\left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n t}\sin\left(\omega_d t+\phi\right)\right]u_{-1}(t) - \left[\left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n (t-t_1)}\sin\left(\omega_d (t-t_1)+\phi\right)\right]\Big\}u_{-1}(t-t_1). \qquad (55)
\end{aligned}
$$

SOLUTION FOR A SAW-TOOTH PULSE

Remote Input $f_k(t)$

Consider finally the case of a shock input, at point *C*, idealized as a sawtooth pulse of respective initial and final slopes σ_1 and σ_2, and with peak and terminal corner points at times t_1 and t_2, respectively:

$$f_k(t)=\sigma_1 t u_{-1}(t)-\sigma_1(t-t_1)u_{-1}(t-t_1)\ +\sigma_2(t-t_1)u_{-1}(t-t_1)-\sigma_2(t-t_2)u_{-1}(t-t_2). \tag{56}$$

Local Input $d(t)$

Substitution of (56) into (11), yields the following:

$$\begin{aligned}\eta_i(t)=e^{-\varsigma_i\omega_{ni}t}\left(A_i\cos\omega_{di}t+B_i\sin\omega_{di}t\right)u_{-1}(t)+(\mu_{ik}/\omega_{di})\left(e^{-\varsigma_i\omega_{ni}t}\sin\omega_{di}t\right)*\left[\sigma_1 t u_{-1}(t)-\sigma_1(t-t_1)u_{-1}(t-t_1)\right.\\ \left.+\sigma_2(t-t_1)u_{-1}(t-t_1)-\sigma_2(t-t_2)u_{-1}(t-t_2)\right].\end{aligned} \tag{57}$$

Analogously to the previous case, let the convolution term be represented by

$$\begin{aligned}\eta_{i,\ \text{forced}}=(\mu_{ik}/\omega_{di})\left(e^{-\varsigma_i\omega_{ni}t}\sin\omega_{di}t\right)*\left[\sigma_1 t u_{-1}(t)-\sigma_1(t-t_1)u_{-1}(t-t_1)\right.\\ \left.+\sigma_2(t-t_1)u_{-1}(t-t_1)-\sigma_2(t-t_2)u_{-1}(t-t_2)\right].\end{aligned} \tag{58}$$

The Laplace transform of the preceding is

$$\mathrm{H}_{i,\ \text{forced}}(s)=\frac{\mu_{ik}}{s^2+2\varsigma_i\omega_{ni}s+\omega_{ni}^2}\cdot\left[\frac{\sigma_1}{s^2}\left(1-e^{-t_1 s}\right)+\frac{\sigma_2}{s^2}\left(e^{-t_1 s}-e^{-t_2 s}\right)\right]. \tag{59}$$

Taking the inverse Laplace transform ([3], p. 655, no. 5; and [4], p. 915, no. 4.1), and adding back the free-response terms leads (after simplification) to the following complete response for the ith modal coordinate of *S*:

$$\begin{aligned}\eta_i(t)=e^{-\varsigma_i\omega_{ni}t}\left(A_i\cos\omega_{di}t+B_i\sin\omega_{di}t\right)u_{-1}(t)+\mu_{ik}\sigma_1\left[\frac{2\varsigma_i}{\omega_{ni}^3}+\frac{1}{\omega_{ni}^2}t+\frac{1}{\omega_{ni}^2\omega_{di}^2}e^{-\varsigma_i\omega_{ni}t}\sin\left(\omega_{di}t+\psi_i\right)\right]u_{-1}(t)\\ +\mu_{ik}\left(\sigma_2-\sigma_1\right)\left[\frac{2\varsigma_i}{\omega_{ni}^3}+\frac{1}{\omega_{ni}^2}(t-t_1)+\frac{1}{\omega_{ni}^2\omega_{di}^2}e^{-\varsigma_i\omega_{ni}(t-t_1)}\sin\left(\omega_{di}(t-t_1)+\psi_i\right)\right]u_{-1}(t-t_1)\\ -\mu_{ik}\sigma_2\left[\frac{2\varsigma_i}{\omega_{ni}^3}+\frac{1}{\omega_{ni}^2}(t-t_2)+\frac{1}{\omega_{ni}^2\omega_{di}^2}e^{-\varsigma_i\omega_{ni}(t-t_2)}\sin\left(\omega_{di}(t-t_2)+\psi_i\right)\right]u_{-1}(t-t_2),\end{aligned} \tag{60}$$

where

$$\psi_i=-2\tan^{-1}\frac{\sqrt{1-\varsigma_i^2}}{-\varsigma_i}. \tag{61}$$

Substituting (60) into (14), the filtered input is found to be

$$\begin{aligned}d(t)=\left[\sum_{i=1}^{n}D_{i1}e^{-\varsigma_i\omega_{ni}t}\cos\omega_{di}t+\sum_{i=1}^{n}D_{i2}e^{-\varsigma_i\omega_{ni}t}\sin\omega_{di}t+\sum_{i=1}^{n}D_{i3}e^{-\varsigma_i\omega_{ni}t}\sin\left(\omega_{di}t+\psi_i\right)+\sum_{i=1}^{n}D_{i6}+\sum_{i=1}^{n}D_{i7}t\right]u_{-1}(t)\\ +\left\{\sum_{i=1}^{n}D_{i5}e^{-\varsigma_i\omega_{ni}(t-t_1)}\sin\left[\omega_{di}(t-t_1)+\psi_i\right]+\sum_{i=1}^{n}D_{i10}+\sum_{i=1}^{n}D_{i11}(t-t_1)\right\}u_{-1}(t-t_1)\\ -\left\{\sum_{i=1}^{n}D_{i4}e^{-\varsigma_i\omega_{ni}(t-t_2)}\sin\left[\omega_{di}(t-t_2)+\psi_i\right]+\sum_{i=1}^{n}D_{i18}+\sum_{i=1}^{n}D_{i19}(t-t_2)\right\}u_{-1}(t-t_2),\end{aligned} \tag{62}$$

or, after rearranging,

$$d(t)=\left[\sum_{i=1}^{n}D_{i1}e^{-\varsigma_i\omega_{ni}t}\cos\omega_{di}t+\sum_{i=1}^{n}D_{i2}e^{-\varsigma_i\omega_{ni}t}\sin\omega_{di}t+\sum_{i=1}^{n}D_{i3}e^{-\varsigma_i\omega_{ni}t}\sin(\omega_{di}t+\psi_i)\right]u_{-1}(t)$$

$$+\left\{\sum_{i=1}^{n}D_{i5}e^{-\varsigma_i\omega_{ni}(t-t_1)}\sin[\omega_{di}(t-t_1)+\psi_i]\right\}u_{-1}(t-t_1)-\left\{\sum_{i=1}^{n}D_{i4}e^{-\varsigma_i\omega_{ni}(t-t_2)}\sin[\omega_{di}(t-t_2)+\psi_i]\right\}u_{-1}(t-t_2)$$

$$+\sum_{i=1}^{n}D_{i6}\,u_{-1}(t)+\sum_{i=1}^{n}D_{i10}\,u_{-1}(t-t_1)+\sum_{i=1}^{n}D_{i18}\,u_{-1}(t-t_2)+\sum_{i=1}^{n}D_{i7}t\,u_{-1}(t)+\sum_{i=1}^{n}D_{i11}(t-t_1)u_{-1}(t-t_1)+\sum_{i=1}^{n}D_{i19}(t-t_2)u_{-1}(t-t_2), \tag{63}$$

where

$$D_{i1}=\tilde{m}_{ij}A_i\,, \tag{64}$$

$$D_{i2}=\tilde{m}_{ij}B_i\,, \tag{65}$$

$$D_{i3}=\frac{\tilde{m}_{ij}\mu_{ik}\sigma_1}{\omega_{ni}^2\omega_{di}^2}\,, \tag{66}$$

$$D_{i4}=\frac{\tilde{m}_{ij}\mu_{ik}\sigma_2}{\omega_{ni}^2\omega_{di}^2}\,, \tag{67}$$

$$D_{i5}=D_{i4}-D_{i3}\,, \tag{68}$$

$$D_{i6}=\frac{2\tilde{m}_{ij}\mu_{ik}\sigma_1\varsigma_i}{\omega_{ni}^3}\,, \tag{69}$$

$$D_{i7}=\frac{\tilde{m}_{ij}\mu_{ik}\sigma_1}{\omega_{ni}^2}\,, \tag{70}$$

$$D_{i8}=\frac{2\tilde{m}_{ij}\mu_{ik}\sigma_2\varsigma_i}{\omega_{ni}^2\omega_{di}^2}\,, \tag{71}$$

and

$$D_{i9}=\frac{\tilde{m}_{ij}\mu_{ik}\sigma_2\varsigma_i}{\omega_{ni}^2}\,. \tag{72}$$

Displacement Response $x(t)$

Note that for this description of shock load, the induced input $d(t)$ is now a linear combination of three types of terms: exponentially decaying sinusoids (the first five summations of (63)), constant terms (the next three summations), and ramp terms (the final three summations). For the sinusoidal terms, the displacement response can be determined, as before, using the results of Reference [1]. For the constant terms, the Laplace transform can be used, as in the previous section applies. The forced response to the final three terms can also be determined via Laplace transforms ([4], p. 915, no. 4.2).

Consider a ramp input term of the general form

$$d_{\text{ramp},j}(t)=\left(\sum_{i=1}^{n}D_i\right)(t-t_j)u_{-1}(t-t_1)\,. \tag{73}$$

(Parentheses have been added to emphasize that the right-hand-side expression is a product of three terms.) Using (15), the corresponding term of the forced response is

$$x_{\text{ramp},\,j}(t)=\left[Ce^{-\varsigma\omega_n t}\sin(\omega_d t+\phi)\right]*\left[\left(\sum_{i=1}^{n}D_i\right)(t-t_j)u_{-1}(t-t_1)\right]. \tag{74}$$

The Laplace transform for the first square-bracketed term is found in Equation (11) of [1]. For the second, the Laplace transform is determined using entry 4 on page 655 of [3], along with the time-translation theorem on page 106 of [3]. The result follows:

$$X_{\text{ramp},\,j}(s)=\left(C\frac{2\varsigma\omega_n s+\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}\right)\cdot\left(\sum_{i=1}^{n}\frac{D_i}{s^2}e^{-t_1 s}\right). \tag{75}$$

Rearranging,

$$X_{\text{ramp},\,j}(s)=\left(C\sum_{i=1}^{n}D_i\right)\frac{2\varsigma\omega_n s+\omega_n^2}{s\left(s^2+2\varsigma\omega_n s+\omega_n^2\right)}\cdot e^{-t_1 s}\,, \tag{76}$$

for which the inverse Laplace transform (again using the time-translation theorem, with [4], p. 915, no. 4.2) is, after simplification,

$$x_{\text{ramp},\,j}(t)=\left(C\sum_{i=1}^{n}D_i\right)\left\{\frac{1}{\omega_d}e^{-\varsigma\omega_n(t-t_j)}\sin\left[\omega_d\left(t-t_j\right)+\Omega\right]\quad+\left(t-t_j\right)\right\}u_{-1}\left(t-t_j\right), \tag{77}$$

where

$$\Omega=\tan^{-1}\left(\frac{2\varsigma\sqrt{1-\varsigma^2}}{1-2\varsigma^2}\right)-2\tan^{-1}\left(\frac{\sqrt{1-\varsigma^2}}{-\varsigma}\right). \tag{78}$$

It is now possible to express the full displacement response to the remote sawtooth input:

$$\begin{aligned}
x(t)=\Big\langle\, & e^{-\varsigma\omega_n t}\left(A\cos\omega_d t+B\sin\omega_d t\right)\\
&+C\sum_{i=1}^{n}D_{i1}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\cos\left(\omega_d t+\phi+\theta_{1i}\right)-\delta_{2i}\cos\left(\omega_d t+\phi+\theta_{2i}\right)\right]\\
&\qquad+e^{-\varsigma_i\omega_{ni}t}\left[\delta_{1i}\cos\left(\omega_{di}t+\phi+\theta_{1i}\right)-\delta_{2i}\cos\left(\omega_{di}t-\phi+\theta_{3i}\right)\right]\Big\}\\
&+C\sum_{i=1}^{n}D_{i2}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\sin\left(\omega_d t+\phi+\theta_{1i}\right)+\delta_{2i}\sin\left(\omega_d t+\phi+\theta_{2i}\right)\right]\\
&\qquad+e^{-\varsigma_i\omega_{ni}t}\left[\delta_{1i}\sin\left(\omega_{di}t+\phi+\theta_{1i}\right)-\delta_{2i}\sin\left(\omega_{di}t-\phi+\theta_{3i}\right)\right]\Big\}\\
&+C\sum_{i=1}^{n}D_{i3}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\sin\left(\omega_d t+\phi+\psi_i+\theta_{1i}\right)+\delta_{2i}\sin\left(\omega_d t+\phi-\psi_i+\theta_{2i}\right)\right]\\
&\qquad+e^{-\varsigma_i\omega_{ni}t}\left[\delta_{1i}\sin\left(\omega_{di}t+\psi_i+\phi+\theta_{1i}\right)-\delta_{2i}\sin\left(\omega_{di}t+\psi_i-\phi+\theta_{3i}\right)\right]\Big\}\\
&+C\sum_{i=1}^{n}D_{i6}\left[1+\left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n t}\sin\left(\omega_d t+\phi\right)\right]\\
&+C\sum_{i=1}^{n}D_{i7}\left(1/\omega_d\right)e^{-\varsigma\omega_n t}\sin\left(\omega_d t+\Omega\right)\Big\rangle u_{-1}(t)\\
+\Big\{ & C\sum_{i=1}^{n}D_{i5}\Big\langle e^{-\varsigma\omega_n(t-t_1)}\left\{-\delta_{1i}\sin\left[\omega_d(t-t_1)+\phi+\psi_i+\theta_{1i}\right]+\delta_{2i}\sin\left[\omega_d(t-t_1)+\phi-\psi_i+\theta_{2i}\right]\right\}\\
&\qquad+e^{-\varsigma_i\omega_{ni}(t-t_1)}\left\{\delta_{1i}\sin\left[\omega_{di}(t-t_1)+\psi_i+\phi+\theta_{1i}\right]-\delta_{2i}\sin\left[\omega_{di}(t-t_1)+\psi_i-\phi+\theta_{3i}\right]\right\}\Big\rangle\\
&+C\sum_{i=1}^{n}D_{i10}\left\{1+\left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n(t-t_1)}\sin\left[\omega_d\left(t-t_1\right)+\phi\right]\right\}\\
&+C\sum_{i=1}^{n}D_{i11}\left\{\left(1/\omega_d\right)e^{-\varsigma\omega_n(t-t_1)}\sin\left[\omega_d\left(t-t_1\right)+\Omega\right]+\left(t-t_1\right)\right\}\Big\}u_{-1}\left(t-t_1\right)\\
-\Big\{ & C\sum_{i=1}^{n}D_{i4}\Big\langle e^{-\varsigma\omega_n(t-t_2)}\left\{-\delta_{1i}\sin\left[\omega_d(t-t_2)+\phi+\psi_i+\theta_{1i}\right]+\delta_{2i}\sin\left[\omega_d(t-t_2)+\phi-\psi_i+\theta_{2i}\right]\right\}\\
&\qquad+e^{-\varsigma_i\omega_{ni}(t-t_2)}\left\{\delta_{1i}\sin\left[\omega_{di}(t-t_2)+\psi_i+\phi+\theta_{1i}\right]-\delta_{2i}\sin\left[\omega_{di}(t-t_2)+\psi_i-\phi+\theta_{3i}\right]\right\}\Big\rangle\\
&+C\sum_{i=1}^{n}D_{i8}\left\{1+\left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n(t-t_2)}\sin\left[\omega_d\left(t-t_2\right)+\phi\right]\right\}
\end{aligned}$$

$$+C\sum_{i=1}^{n}D_{i9}\left\{\left(1/\omega_d\right)e^{-\varsigma\omega_n(t-t_2)}\sin\left[\omega_d\left(t-t_2\right)+\Omega\right]+\left(t-t_1\right)\right\}\Bigg\}u_{-1}\left(t-t_2\right), \tag{79}$$

where $$D_{i10}=D_{i8}-D_{i6}\,, \tag{80}$$

and $$D_{i11}=D_{i9}-D_{i7}\,. \tag{81}$$

All other variables have been previously defined.

Relative Displacement $\delta(t)$

Subtracting (63) from (79), one finally obtains an expression for the relative displacement:

$$\begin{aligned}
\delta(t)=x(t)-d(t)\ &=\Big\langle e^{-\varsigma\omega_n t}\left(A\cos\omega_d t+B\sin\omega_d t\right)\\
&+C\sum_{i=1}^{n}D_{i1}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\cos\left(\omega_d t+\phi+\theta_{1i}\right)-\delta_{2i}\cos\left(\omega_d t+\phi+\theta_{2i}\right)\right]\\
&\qquad+e^{-\varsigma_i\omega_{ni}t}\left[\delta_{1i}\cos\left(\omega_{di}t+\phi+\theta_{1i}\right)-\delta_{2i}\cos\left(\omega_{di}t-\phi+\theta_{3i}\right)-\left(1/C\right)\cos\omega_{di}t\right]\Big\}\\
&+C\sum_{i=1}^{n}D_{i2}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\sin\left(\omega_d t+\phi+\theta_{1i}\right)+\delta_{2i}\sin\left(\omega_d t+\phi+\theta_{2i}\right)\right]\\
&\qquad+e^{-\varsigma_i\omega_{ni}t}\left[\delta_{1i}\sin\left(\omega_{di}t+\phi+\theta_{1i}\right)-\delta_{2i}\sin\left(\omega_{di}t-\phi+\theta_{3i}\right)-\left(1/C\right)\sin\omega_{di}t\right]\Big\}\\
&+C\sum_{i=1}^{n}D_{i3}\Big\{e^{-\varsigma\omega_n t}\left[-\delta_{1i}\sin\left(\omega_d t+\phi+\psi_i+\theta_{1i}\right)+\delta_{2i}\sin\left(\omega_d t+\phi-\psi_i+\theta_{2i}\right)\right]\\
&\qquad+e^{-\varsigma_i\omega_{ni}t}\left[\delta_{1i}\sin\left(\omega_{di}t+\psi_i+\phi+\theta_{1i}\right)-\delta_{2i}\sin\left(\omega_{di}t+\psi_i-\phi+\theta_{3i}\right)-\left(1/C\right)\sin\omega_{di}\left(t+\psi_i\right)\right]\Big\}\\
&+C\sum_{i=1}^{n}D_{i6}\left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n t}\sin\left(\omega_d t+\phi\right)+C\sum_{i=1}^{n}D_{i7}\left(1/\omega_d\right)e^{-\varsigma\omega_n t}\sin\left(\omega_d t+\Omega\right)\Big\rangle u_{-1}(t)\\
&+\Bigg\{C\sum_{i=1}^{n}D_{i5}\Big\langle e^{-\varsigma\omega_n(t-t_1)}\left\{-\delta_{1i}\sin\left[\omega_d(t-t_1)+\phi+\psi_i+\theta_{1i}\right]+\delta_{2i}\sin\left[\omega_d(t-t_1)+\phi-\psi_i+\theta_{2i}\right]\right\}\\
&\qquad+e^{-\varsigma_i\omega_{ni}(t-t_1)}\left\{\delta_{1i}\sin\left[\omega_{di}(t-t_1)+\psi_i+\phi+\theta_{1i}\right]-\delta_{2i}\sin\left[\omega_{di}(t-t_1)+\psi_i-\phi+\theta_{3i}\right]-\left(1/C\right)\sin\left[\omega_{di}(t-t_1)+\psi_i\right]\right\}\Big\rangle\\
&+C\sum_{i=1}^{n}D_{i10}\left\{\left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n(t-t_1)}\sin\left[\omega_d\left(t-t_1\right)+\phi\right]\right\}\\
&+C\sum_{i=1}^{n}D_{i11}\left\{\left(1/\omega_d\right)e^{-\varsigma\omega_n(t-t_1)}\sin\left[\omega_d\left(t-t_1\right)+\Omega\right]+\left(t-t_1\right)\right\}\Bigg\}u_{-1}\left(t-t_1\right)\\
&-\Bigg\{C\sum_{i=1}^{n}D_{i4}\Big\langle e^{-\varsigma\omega_n(t-t_2)}\left\{-\delta_{1i}\sin\left[\omega_d(t-t_2)+\phi+\psi_i+\theta_{1i}\right]+\delta_{2i}\sin\left[\omega_d(t-t_2)+\phi-\psi_i+\theta_{2i}\right]\right\}\\
&\qquad+e^{-\varsigma_i\omega_{ni}(t-t_2)}\left\{\delta_{1i}\sin\left[\omega_{di}(t-t_2)+\psi_i+\phi+\theta_{1i}\right]-\delta_{2i}\sin\left[\omega_{di}(t-t_2)+\psi_i-\phi+\theta_{3i}\right]-\left(1/C\right)\sin\left[\omega_{di}(t-t_2)+\psi_i\right]\right\}\Big\rangle\\
&+C\sum_{i=1}^{n}D_{i8}\left\{\left(\omega_n/\omega_d\right)e^{-\varsigma\omega_n(t-t_2)}\sin\left[\omega_d\left(t-t_2\right)+\phi\right]\right\}\\
&+C\sum_{i=1}^{n}D_{i9}\left\{\left(1/\omega_d\right)e^{-\varsigma\omega_n(t-t_2)}\sin\left[\omega_d\left(t-t_2\right)+\Omega\right]+\left(t-t_1\right)\right\}\Bigg\}u_{-1}\left(t-t_2\right).
\end{aligned} \tag{82}$$

CONCLUSION

This paper has presented the analytical time response, of the (hypothetical) MSD system used for SRS determination, for a remote shock input of three forms: an ideal impulse, a rectangular pulse, and a saw-tooth pulse. The response in each case is given as an absolute and as a relative displacement. Having these analytical expressions for the kinematic quantities underlying the various SRS's permits the SRS's to be computed exactly for

these remote shock inputs, to a linear system (assuming knowledge of the system's eigenstructure), without necessitating numerical evaluation of a convolution integral. The equations can be used as a benchmark to evaluate the accuracy of other methods of SRS determination. They can also be used to determine the minimum number of modes required, in a system's finite-element model, to produce an SRS of specified accuracy.

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