Geometric Control of Quantum Systems–Progress Report

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3 November 2004

Overview

1. Quantum Computation and Quantum Control
2. Example: NMR Quantum Computers
3. Griffiths formalism
4. Progress

Quantum Computation and Quantum Control

- Objective: create quantum machines with unusual capabilities
- General obstacle: *decoherence* of quantum systems
- Need: optimal control of quantum systems with high precision

The optimal control problem

Let $|\psi(t)> \in \mathbb{C}^{2^n}$ be the state vector for an n-qubit system at time t.

Goal: transition from the initial state $|\psi_0>$ to a desired final state $|\psi_f>= U_f\,|\psi_0>$ in the shortest time possible (to minimize decoherence).

The optimal control problem, cont'd

$$|\psi(t)> = U(t)|\psi_0>$$

where $U(t)$ is a solution curve in $SU(2^n)$ of

$$\dot{U}(t) = -\frac{i}{\hbar}H(t)U(t),$$
$$U(0) = \mathbf{1}.$$

There are many Hamiltonians $H(t)$ which will permit $U(t)$ to evolve to U_f in finite time. The optimal control problem becomes: can we control $H(t)$ so that the curve $U(t)$ is as short as possible?

Example of a curve in $SU(4)$

$$U(t) = \begin{bmatrix} \cos(t/2) & -\sin(t/2) & 0 & 0 \\ \sin(t/2) & \cos(t/2) & 0 & 0 \\ 0 & 0 & \cos(t/2) & -\sin(t/2) \\ 0 & 0 & \sin(t/2) & \cos(t/2) \end{bmatrix}.$$

This transforms $|\psi_0>= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ into $|\psi_f>= \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ as $0 \leq t \leq \pi$.

Example : control of NMR spin systems

- Khaneja, Brockett and Glaser: the optimal control problem for NMR reduces to finding sub-Riemannian geodesics in $SU(2^n)$ *PRA* **65:** 032301 (2002)
- their results apply directly to systems of 2 or 3 qubits
- is there a method for extending their solution to systems with $n > 3$ qubits?

The Hamiltonian of an n-spin NMR system can be decomposed as

$$H = H_d + \sum_{j=1}^{n} u_j H_j \tag{1}$$

where H_d is the *drift* Hamiltonian (internal couplings), the u_j are *controls* and the H_j are the *rf* or *control* Hamiltonians.

The control subbasis $\{iH_j\}$ generates the Lie algebra of the closed Lie subgroup $K = \bigotimes_{j=1}^{n} SU(2) \subset SU(2^n)$.

Product operator basis for the Lie algebra $su(2^n)$

The *product operator* basis for $su(2^n)$ is $\{iB_s\}$ where

$$B_s = 2^{q-1} \prod_{k=1}^{n} (I_{k\alpha})^{a_{ks}},$$

$$\alpha \in \{\, x, y, z \,\}$$

$$I_{k\alpha} = \mathbf{1} \otimes \cdots \otimes \sigma_\alpha \otimes \mathbf{1}$$

and $1 \leq q \leq n$.

Accessible directions in the NMR control problem

The set of possible velocity vectors (*accessible directions*) in $SU(2^n)$ is

$$Ad_K\,(iH_d) = \{\, k(iH_d)\,k^{-1} \mid k \in K \,\}.$$

As before, $K = SU(2) \otimes \cdots \otimes SU(2)$ (n factors).

This is a proper subspace of the entire tangent space at any point. (The velocity vectors are subject to a non-holonomic constraint.) However, this subspace is *bracket-generating*.

Bracket-generating subspaces

A subspace D of the Lie algebra is *bracket-generating* if the entire Lie algebra is generated by iterated Lie brackets of elements of D.

Chow (1939): if D is bracket-generating, any two points in the Lie group G can be joined by a piecewise-smooth curve $U : [a, b] \to G$ so that $\dot{U}(t) \in D$ whenever $\dot{U}(t)$ is defined.

In the language of control theory, the reachable set is the entire Lie group $G = SU(2^n)$.

Sub-Riemannian geometry

A positive-definite inner product, called a *sub-Riemannian metric* can be defined on the subspace $D = Ad_K(H_d)$ of the Lie algebra. Since all the control curves $U(t)$ are tangent to this subspace, we can define a length functional on these curves by the formula

$$L(U) = \int_a^b \langle \dot{U}(t), \dot{U}(t) \rangle^{1/2} \, dt. \tag{2}$$

Solution curves $U(t)$ that minimize this length functional are *sub-Riemannian geodesics.*

Constrained variational calculus

The problem of finding optimal controls for the NMR n-qubit system thus reduces to finding sub-Riemannian geodesics in $SU(2^n)$. Khaneja et al solved this problem for the 2- and 3-qubit case directly by using special properties of $SU(4)$ and $SU(8)$, but this method does not directly apply to the case of $n > 3$ qubits.

The problem of finding sub-Riemannian geodesics is a variational problem with linear velocity constraints. There exists a general method for computing these variations, called the Griffiths formalism (Griffiths 1983).

Griffiths formalism for constrained variations

Let $X = D \times \mathbb{R}$ and let ϕ be the *energy 1-form*

$$\phi = \frac{1}{2}((p^1)^2 + (p^2)^2 + \cdots + (p^k)^2)\, dt \tag{3}$$

on X. Let $Z \subset T^*X$ be the submanifold defined by

$$Z = \bigcup_{x \in X} Z_x$$

where $Z_x = \{\phi(x) + I_x \subset T_x^*X\}$, and I is the defining coframing of D lifted to X.

Griffiths formalism, cont'd

The constrained variational problem on the state space M is thus lifted to an unconstrained variational problem on Z.

The integral curves of the Cartan system of the canonical symplectic 2-form on Z project to regular sub-Riemannian geodesics on the state space M.

Griffiths formalism, cont'd

This method was used in the speaker's Ph.D. thesis to find sub-Riemannian geodesics on Engel four-manifolds. In particular, explicit solutions were found for sub-Riemannian geodesics on the Lie groups $SO(3) \times S^1$, $SEuc(2) \times S^1$, and $SO(2,1) \times S^1$.

More recently this method was applied to compute geodesics of sub-Finsler three-manifolds (Clelland, M, math.DG/0406439).

- **Advantage:**
  - applicable to quantum control problems with any finite number of spins
- **Disadvantage:** number of differential equations becomes rapidly larger as the number of spins increases

Computational challenges

The Griffiths formalism yields ODEs based on the structure equations of the Lie group. The number of structure equations grows rapidly with the number n of qubits.

n	unitary group	number of structure eqns
1	$SU(2)$	3
2	$SU(4)$	15
3	$SU(8)$	63
4	$SU(16)$	255
5	$SU(32)$	1023

Progress so far

- 2- and 3-qubit cases done by Khaneja et al (*PRA* **65:** 032301 (2002)) using different methods
- 4-qubit case underway, will take many man-hours

So, why do it?

1) Mitigating factors:
   a) structure equations for $SU(2^n)$ have similar characteristics for all n;
   b) computer algebra software can be adapted.
2) This optimization problem exists with the same general outlines for all quantum computation schemes, not just NMR.

Summary:

- it is possible to extend Khaneja et al's results, with the help of differential geometric methods as outlined above.
- the control problem, with different specifics, is relevant to other quantum computation systems
- the computations will be nontrivial
- the possible rewards are worthwhile, since control of substantial numbers of qubits is required for useful quantum information processing