Using Lie Symmetries to Study Differential Equations

Rachelle C. DeCoste[†] and Louis Piscitelle[‡]

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[†]Department of Mathematical Sciences
United States Military Academy
West Point, New York

[‡]U.S. Army RDECOM
Natick Soldier Center
Natick, Massachusetts

ABSTRACT

Lie symmetries of differential equations are diffeomorphisms which map a solution of the given equation to another solution. There are many applications of symmetric groups to differential equations. Most of the well-known techniques for solving differential equations are special cases of a few symmetry methods. We can use symmetries to find exact solutions of differential equations. In addition, using known solutions and a symmetry group, one can compute new solutions for a differential equation.

Two applications we will discuss are the use of symmetry groups to find explicit group-invariant solutions and the classification of differential equations by their symmetry groups. We use the first of these applications to compare invariant solutions arising through symmetries with invariant solutions, such as center manifolds, which are found with the use of dynamical systems theory. The consideration of classifying differential equations according to their symmetry groups may be directly applicable to those differential equations which arise in the work of scientists at the Natick Soldier Center. We hope future work on classification will allow the researchers an easier path to finding a solution to a system of differential equations outside of the traditional methods that have been employed. In addition, this research will enable enhanced interaction with respect to the solving of various applied mathematics problems at the Natick Soldier Center.

We present preliminary results on Lie symmetries of differential equations employed in the research of scientists at the Natick Soldier Center. One example of particular interest is a model used to characterize and predict microbial growth-decay kinetics in foods.

1. Introduction

This article introduces two important concepts in the study of differential equations and dynamical systems. First considered are Lie symmetries, which are diffeomorphisms mapping the set of solutions of a differential equation to itself. These symmetries give rise to methods of solving differential equations. Additionally, they lead to a classification of differential equations by the symmetries acting on the given equations. One important outcome of these symmetries is the set of manifolds that are invariant under the action of

the symmetry group on a differential equation. Thus the other main study of this paper concerns invariant manifolds which arise in the study of a dynamical system, namely the stable, unstable and center manifolds. One way to link the two ideas is to ask how a Lie symmetry of a system acts on these manifolds. Stable and unstable manifolds will remain invariant under the symmetry action, while the action on the center manifold is more complicated. Some recent results from the literature on symmetry actions on center manifolds are discussed. Another important link between the study of Lie symmetries and the dynamics of a system of differential equations is that they both provide ways to investigate properties of a system by reducing to a manifold of dimension smaller than that of the original system. As discussed below, differential invariants of Lie symmetries and center manifolds both provide ways to accomplish this. In future work we hope to apply these theoretical results to systems of equations arising in the research of scientists at the Natick Soldier Research Center. This paper concludes with an example of one such system that we will consider with respect to these ideas.

2. Lie Symmetries

Lie symmetries are diffeomorphisms mapping a set of solutions of a given differential equation to itself. Consider a differential equation $y^{(n)} = \omega(x, y, y', \ldots, y^{(n-1)})$ where $y^{(k)} \equiv \dfrac{d^k y}{dx^k}$ and ω is assumed smooth. Next consider a diffeomorphism $\Gamma : (x, y) \mapsto (\hat{x}, \hat{y})$. Let $\hat{y}^{(k)} = \frac{D_x \hat{y}^{(k-1)}}{D_x \hat{x}}$, with $D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots$ the total derivative in x. Then a diffeomorphism is a Lie symmetry if it maps a solution of a differential equation to another solution. Mathematically, this is equivalent to $\hat{y}^{(n)} = \omega(\hat{x}, \hat{y}, \hat{y}', \ldots, \hat{y}^{(n-1)})$ when $y^{(n)} = \omega(x, y, y', \ldots, y^{(n-1)})$. This is called the **symmetry condition**. We illustrate how this condition is used for symmetries on the plane through the following examples.

Example 2.1. *Consider* $\dfrac{dy}{dx} = \dfrac{2y}{x}$ *with* $(\hat{x}, \hat{y}) = (e^{\varepsilon}x, e^{-2\varepsilon}y)$. *To show that* Γ *defined by* $(\hat{x}, \hat{y})$ *is a Lie symmetry, we need to show that* $\dfrac{d\hat{y}}{d\hat{x}} = \dfrac{2\hat{y}}{\hat{x}}$. *We calculate* $\dfrac{d\hat{y}}{d\hat{x}} = \dfrac{D_x\hat{y}}{D_x\hat{x}} = \dfrac{e^{-2\varepsilon}y'}{e^{\varepsilon}} = \dfrac{e^{-2\varepsilon}}{e^{\varepsilon}}\dfrac{2y}{x} = \dfrac{2\hat{y}}{\hat{x}}$. *Thus this mapping defines a nontrivial one-parameter Lie group action on the given differential equation. A solution curve* $y = c_1x^2$ *is mapped to the curve defined by* $(\hat{x}, \hat{y}) = (e^{\varepsilon}x, c_1e^{-\varepsilon}x^2)$, *or* $\hat{y} = c_1e^{-3\varepsilon}\hat{x}^2$.

An example of a trivial action on a differential equation is the Lie group $(\hat{x}, \hat{y}) = (x + \varepsilon, y)$ acting on $y' = 0$ since the solution curves for the equation are the lines $y = c$ for some constant c and the action simply shifts a point on the line to another point on the same line.

Geometrically a Lie symmetry relates to the tangent space of the orbits of points under the action of the symmetry. Consider an ODE on the plane and define $\xi(x, y) = \left.\dfrac{d\hat{x}}{d\varepsilon}\right|_{\varepsilon=0}$ and $\eta(x, y) = \left.\dfrac{d\hat{y}}{d\varepsilon}\right|_{\varepsilon=0}$. Then $(\xi(x, y), \eta(x, y))$ is the tangent vector to the orbit under

the Lie symmetry at the point (x, y) on a solution curve. These terms $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ are called **infinitesimals** and the operator $X = \xi\partial_x + \eta\partial_y$ is called an **infinitesimal generator** of the Lie symmetry given by $(\hat{x}, \hat{y})$. If we can find the infinitesimals of the symmetry, then we can find the symmetry itself as in the following example.

Example 2.2. *Suppose that we have found that a given differential equation has a symmetry with infinitesimal generator $X = \partial_x + y\partial_y$. We calculate the one-parameter Lie group associated with this generator. Our infinitesimals $(\xi, \eta) = (1, y)$ give $\frac{d\hat{x}}{d\varepsilon} = 1$, and $\frac{d\hat{y}}{d\varepsilon} = \hat{y}$ for $\varepsilon = 0$. We solve these equations with the initial conditions that at $\varepsilon = 0$, $(\hat{x}(x, y), \hat{y}(x, y)) = (x, y)$. Thus $\hat{x}(x, y; \varepsilon) = \varepsilon + A(x, y) = \varepsilon + x$ and $\hat{y}(x, y; \varepsilon) = B(x, y)e^{\varepsilon} = ye^{\varepsilon}$. Our one-parameter Lie group of symmetries is then $(\hat{x}, \hat{y}) = (\varepsilon + x, ye^{\varepsilon})$.*

We extend these ideas of one-parameter symmetries in the plane to Lie symmetries with more than one parameter and to higher order differential equations. We can naturally extend a diffeomorphism $\Gamma : (x, y) \mapsto (\hat{x}, \hat{y})$ in the plane to a diffeomorphism in the $(n + 2)$-dimensional Euclidean space with variables $x, y, y', \ldots, y^{(n)}$. This extension is given by:

$$\Gamma : (x, y, y', \ldots, y^{(n)}) \mapsto (\hat{x}, \hat{y}, \hat{y}', \ldots, \hat{y}^{(n)})$$

with $\hat{y}^{(k)} = \dfrac{D_x\hat{y}^{(k-1)}}{D_x\hat{x}}$ as defined above. This map is called the nth **prolongation** of Γ. We use this prolongation along with the symmetry condition to find Lie symmetries for a given differential equation. We can linearize both the symmetry condition and the prolongation by considering the Taylor expansions in the parameter ε. For the diffeomorphism, we obtain:

$$\begin{aligned} \hat{x} &= x + \varepsilon\xi + O(\varepsilon^2) \\ \hat{y} &= y + \varepsilon\eta + O(\varepsilon^2) \\ \hat{y}^{(k)} &= y^{(k)} + \varepsilon\eta^{(k)} + O(\varepsilon^2). \end{aligned}$$

Thus we have $\hat{y}^{(n)} = y^{(n)} + \varepsilon\eta^{(n)} + O(\varepsilon^2)$. From the symmetry condition $\hat{y}^{(n)} = \omega(\hat{x}, \hat{y}', \ldots, \hat{y}^{(n-1)})$, we obtain $\hat{y}^{(n)} = \omega + \varepsilon D_x\omega + O(\varepsilon^2)$. By equating the ε terms, we find

$$\eta^{(n)} = \xi\omega_x + \eta\omega_y + \eta^{(1)}\omega_{y'} + \cdots + \eta^{(n-1)}\omega_{y^{(n-1)}}. \tag{1}$$

This is the **linearized symmetry condition**, which means that if this equality holds for all x and y, then ξ and η are infinitesimals of our Lie symmetry.

From $\hat{y}^{(k)}$, we find the terms $\eta^{(k)}$. These terms are calculated recursively by

$$\eta^{(k)}(x, y, y', \ldots, y^{(k)}) = D_x\eta^{(k-1)} - y^{(k)}D_x\xi.$$

We list the first few:

$$\begin{aligned}
\eta^{(1)} &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2; \\
\eta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \\
&\quad + \{\eta_y - 2\xi_x - 3\xi_y y'\}y''; \\
\eta^{(3)} &= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 \\
&\quad - \xi_{yyy}y'^4 + 3\{\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy}y'^2\}y'' \\
&\quad - 3\xi_y y''^2 + \{\eta_y - 3\xi_x - 4\xi_y y'\}y'''.
\end{aligned}$$

We can use the prolongation of the infinitesimal generator to rewrite the linearized symmetry condition. We find $X^{(n)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \eta^{(2)}\partial_{y^{(2)}} + \cdots + \eta^{(n)}\partial_{y^{(n)}}$. Then (1) is equivalent to

$$X^{(n)}\left(y^{(n)} - \omega(x, y, y', \ldots, y^{(n-1)})\right) = 0. \tag{2}$$

Thus if we can find the infinitesimals ξ and η such that equation (2) holds, we can find the symmetry $\Gamma : (x, y) \mapsto (\hat{x}, \hat{y})$ of a differential equation as in the example above.

Example 2.3. *The symmetries of the differential equation $y'' = 0$.*

In this case we find $\Gamma : (x, y) \mapsto (\hat{x}, \hat{y})$ such that $\hat{y}'' = 0$ when $y'' = 0$. We begin as usual with the infinitesimal generator $X = \xi\partial_x + \eta\partial_y$. Using the linearized symmetry condition $X^{(2)}(y'') = 0$, we find the necessary condition on ξ and η to be $\eta^{(2)} = 0$. Then the following must hold:

$$\begin{aligned}
\eta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \\
&\quad + \{\eta_y - 2\xi_x - 3\xi_y y'\}y'' \\
&= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \ (\textit{since } y'' = 0) \\
&= 0
\end{aligned}$$

Since ξ and η are functions of x and y, we look at this polynomial in the derivative of y and set each coefficient equal to zero. This results in a series of partial differential equations. These PDEs are called ***determining equations*** *for the Lie symmetry.*

In this case we have the following determining equations:

$$\eta_{xx} = 0; \tag{1}$$

$$2\eta_{xy} - \xi_{xx} = 0; \tag{2}$$

$$\eta_{yy} - 2\xi_{xy} = 0; \tag{3}$$

$$\xi_{yy} = 0 \tag{4}$$

From (4), we see that $\xi(x, y) = A(x)y + B(x)$. Then from (3) we find $\eta_{yy} = 2A'(x)$ and thus $\eta(x, y) = A'(x)y^2 + C(x)y + D(x)$. Using these results, (2) gives $4A''(x)y + 2C'(x) = A''(x)y + B''(x)$ which must hold for all values of y. Equating coefficients

of y and constant terms, we find $A''(x) = 0$, thus $A(x) = \alpha_1 x + \alpha_2$ and $2C'(x) = B''(x)$. Finally, from (4) we find $C''(x) = 0$ and $D''(x) = 0$, thus $C(x) = \alpha_3 x + \alpha_4$ and $D(x) = \alpha_5 x + \alpha_6$. Returning to the relationship between $B(x)$ and $C(x)$ from (2), we find $B(x) = \alpha_3 x^2 + \alpha_7 x + \alpha_8$. Thus our infinitesimals are $\xi(x, y) = \alpha_1 xy + \alpha_2 y + \alpha_3 x^2 + \alpha_7 x + \alpha_8$ and $\eta(x, y) = \alpha_1 y^2 + \alpha_3 xy + \alpha_4 y + \alpha_5 x + \alpha_6$. We have found that the differential equation $y'' = 0$ has an eight-parameter Lie group of symmetries. Thus we have 8 different one-parameter groups defined by the infinitesimal generators as follows:

$$\begin{array}{ll} X_1 = xy & X_5 = x \\ X_2 = y\partial_x & X_6 = \partial_y \\ X_3 = x^2\partial_x + xy\partial_y & X_7 = x\partial_x \\ X_4 = y\partial_y & X_8 = \partial_x \end{array}$$

Considering X_8, we see that the one-parameter Lie group associated to this infinitesimal generator is the group defined by the map $\Gamma : (x, y) \mapsto (x + \varepsilon, y)$ since $\xi_8 = 1$ and $\eta_8 = 0$. We can similarly find $\hat{x}_i, \hat{y}_i$ for $i = 1, \ldots, 7$.

The set of infinitesimal generators forms a Lie algebra. In the example above, we have an eight dimensional Lie algebra with basis $\{X_i\}$, $i = 1, \ldots, 8$. In general, the dimension of the Lie algebra is constrained by the order of the ODE (see [H] p 51). Recall that a Lie algebra has a bracket structure where the bracket is an operator such that $[X_i, X_j] = X_i X_j - X_j X_i$. The Lie bracket satisfies the following:

(1) Bilinearity:
$[aX_i + bX_j, X_k] = a[X_i, X_k] + b[X_j, X_k], \quad [X_i, aX_j + bX_k] = a[X_i, X_j] + b[X_i, X_k]$
(2) Skew symmetry: $[X_i, X_j] = -[X_j, X_i]$
(3) Jacobi Identity: $[X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]] = 0$.

In the example above we can calculate all the brackets and thus determine the Lie algebra. Just considering $[X_1, X_i]$, we see $[X_1, X_4] = -X_1$ and $[X_1, X_5] = -X_3$ while the bracket is zero for all other X_i. If we had another differential equation admitting an eight dimensional Lie algebra, we would be able to determine if the two equations lie in the same class according to Lie symmetry classifications. Two differential equations with the same infinitesimal generators (under a canonical basis) are defined to be in the same class of equations. Thus classifying Lie algebras arising from Lie symmetries gives a classification of the differential equations associated to these Lie algebras. According to Hydon ([H] p 176), the classification has been done for scalar ODEs, but not for PDEs or systems of ODEs, which must be handled on a case-by-case basis. This is something we will investigate further in later work.

In his work of 1883, Lie classified arbitrary order ordinary differential equations. For example in his consideration of second order ODEs he found that they admit Lie algebras of dimension 1, 2, 3, 4, or 8 only. Considering only second order ODEs that are associated with two-dimensional Lie algebras, Lie found that there are four canonical second order ODEs. In terms of canoncial coordinates, the four types of differential equations are:

(1) $y'' = f(y')$
(2) $y'' = f(x)$
(3) $y'' = \frac{1}{x} f(y')$
(4) $y'' = f(x)y'$

The derivation of these canonical differential equations can be found in [I] along with a discussion on the higher dimensional Lie algebras associated to second order ODEs. In future work we aim to classify the differential equations used in the research of scientists at the Natick Soldier Center according to their Lie symmetries. We hope to find that there are connections among equations used in the research in different areas based on these symmetries.

3. Invariant Solutions

As we have discussed, Lie symmetries act on solutions of differential equations by mapping the set of solutions to itself. One set of solutions that are of particular interest are those which remain invariant under the action of a Lie symmetry. Let us focus our discussion of equations in the plane, as these ideas are extendable to higher dimensions. Under the action of a one-parameter Lie group, any point (x, y) on a solution curve C defined by $y = f(x)$ is mapped to another point $(\hat{x}, \hat{y})$ on another (possibly the same) solution curve. If for every point (x, y) on C, $(\hat{x}, \hat{y})$ is also on C, we say that $y = f(x)$ is an invariant solution. In this case, the orbit of the point (x, y) will lie in the curve C. If a solution is not an invariant solution, the orbit of a point on C and the tangent vector to the curve C at that point will be transverse. Recall that the tangent vector to the orbit at a point (x, y) on a solution curve is given by $(\xi(x, y), \eta(x, y))$. Thus for a given solution curve (x, y), with tangent vector $(1, y')$, the curve is invariant if at each point (x, y) on the curve, the vectors $(1, y')$ and $(\xi(x, y), \eta(x, y))$ are not transverse. Mathematically, this is equivalent to the following condition. A solution curve C given by $y = f(x)$ is invariant under a Lie symmetry if $Q(x, y, y') = 0$ on C, where $Q(x, y, y') = \eta(x, y) - y'\xi(x, y)$. For more details see [H] p 19.

Thus given any one-parameter Lie group acting on a differential equation, we should be able to find all invariant solutions. If a differential equation admits an r-parameter Lie group, each of the infinitesimal generators may have associated invariant solutions. It can then be investigated as to whether an invariant solution under one generator is invariant under all generators. By considering these infinitesimal generators themselves we do not exhaust the list of possible invariant solutions. If we take a combination of these generators with one parameter, we again have a one-parameter Lie group acting on the differential equation. Different invariant solutions may arise in these cases. Thus there may be many invariant solutions and we can use Lie algebras to classify the invariant solutions. This will be investigated further in future work. See [H] Chapter 10 for more details.

Example 3.1. *According to Carr ([Ca], p 177), the differential equation* $\dfrac{dy}{dx} = \dfrac{y^3 + x^2y - y - x}{xy^2 + x^3 + y - x}$ *is invariant under the rotation group, given by* $X = -y\partial_x + x\partial_y$. *We find an invariant solution of the differential equation. In this case,* $Q(x, y, y') = \eta(x, y) - y'\xi(x, y) = x + yy' = 0$ *for an invariant solution. Solving the equation* $y' = \dfrac{-x}{y}$ *gives* $x^2 + y^2 = c$ *for some constant* c. *Returning to the original differential equation and making this substitution, we find that* $\dfrac{dy}{dx} = \dfrac{(c-1)y - x}{(c-1)x + y}$, *which will equal* $y' = \dfrac{-x}{y}$ *if* $c = 1$. *Thus* $y = \pm\sqrt{1 - x^2}$ *is a solution to the original differential equation which is invariant under rotations.*

4. Differential Invariants

In the previous section we discussed solutions that are invariant under the action of a Lie group of symmetries. In this section we consider functions which leave the differential equations invariant under the action of a Lie symmetry. The value of these functions is that they allow us to reduce the order of a given differential equation, which will hopefully aid in finding a solution of the original equation.

Let X be an infinitesimal generator for the Lie symmetries of a given ODE $y^{(n)} = \omega(x, y, y', \ldots, y^{(n-1)})$. The **differential invariants** of the group generated by X are those nonconstant functions that are invariant under X and the prolongations of X. A nonconstant function I is called a kth **order differential invariant** if $X^{(k)}I(x, y, y', \ldots, y^{(k)}) = 0$.

Example 4.1. *The differential invariants of the group generated by* $X = x\partial_x - y\partial_y$. *In this case* $\xi = x$, $\eta = -y$ *and* $\eta^{(1)} = -2y'$. *The first fundamental differential invariant* $r(x, y)$ *is a first integral of* $\dfrac{dx}{x} = \dfrac{dy}{-y}$. *This is a simple separable differential equation with solution* $\ln|x| = -\ln|y| + c$. *Then we let* $r(x, y) = xy$. *Then using this, we find* $v(x, y, y')$, *the second fundamental differential invariant, which is a first integral of* $\dfrac{dx}{x} = \dfrac{dy}{-y} = \dfrac{dy'}{\eta^{(1)}} = \dfrac{dy'}{-2y'}$. *Solving* $\dfrac{dx}{x} = \dfrac{dy'}{-2y'}$ *gives* $\ln|x| = -\frac{1}{2}\ln|y'| + c$, *resulting in a choice of* $v(x, y, y') = x^2y'$. *We write* $I = F(r(x, y), v(x, y, y')) = F(xy, x^2y')$. *Thus any function invariant under the given symmetry is a function of* $r(x, y)$ *and* $v(x, y, y')$.

We can find differential invariants of ODEs with $R-$parameter Lie groups of symmetries similarly. Let $X_1, X_2, \ldots, X_R$ be the set of independent infinitesimal generators for a given ODE with $X_i = \xi_i\partial_x + \eta_i\partial_y$. In this case a nonconstant function

$I(x, y, y', \ldots, y^{(n-1)})$ is a differential invariant if it satisfies the following matrix equation:

$$\begin{pmatrix} \xi_1 & \eta_1 & \eta_1^{(1)} & \cdots & \eta_1^{(R)} \\ \xi_2 & \eta_2 & \eta_2^{(1)} & \cdots & \eta_2^{(R)} \\ \vdots & \vdots & \vdots & & \vdots \\ \xi_R & \eta_R & \eta_R^{(1)} & \cdots & \eta_R^{(R)} \end{pmatrix} \begin{pmatrix} I_x \\ I_y \\ I_{y'} \\ \vdots \\ I_{y^{(R)}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{5}$$

Using this equation we are always able to find the **fundamental differential invariants** r_R and v_R from which all higher order differential invariants are calculated. By Gaussian elimination we first simplify the matrix equation and then we use the relationships given by the equation to reduce to two relationships which define the fundamental differential invariants. Then we find the higher order differential invariants $v_R^{(1)}, v_R^{(2)}, \ldots, v_R^{(n-1-R)}$, where $v_R^{(1)} = \dfrac{dv_R}{dr_R} = \dfrac{D_x v_R}{D_x r_R}$ and $v_R^{(k)} = \dfrac{d^k v_R}{dr_R^k}$ for any k.

Knowing the differential invariants for a group of transformations allows us to find all differential equations with the given symmetry group. If r_R and v_R are the fundamental differential invariants of an $R-$dimensional Lie group G, then every ODE with order $n \geq R$ with G as its symmetry group, including the given one, can be written in terms of the differential invariants of the group $v_R^{(n-R)} = F(r_R, v_R, \ldots, v_R^{(n-1-R)})$ for some function F. We write the above in terms of $x, y, \ldots, y^{(n)}$ and find a family of ODEs that all have G as a symmetry group. It should be noted that some of these ODEs might have additional symmetries as well.

Example 4.2. *A 3-parameter Lie group of symmetries (*[H] *p 77). The ODE* $y^{(iv)} = \frac{2}{y}(1-y')y'''$ *has the 3-parameter group of Lie symmetries with generators* $X_1 = \partial_x,\ X_2 = x\partial_x + y\partial_y,\ X_3 = x^2\partial_x + 2xy\partial_y$. *From these generators we calculate the fundamental differential invariants by solving*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ x & y & 0 & -y'' & -2y''' \\ x^2 & 2xy & 2y & (2y' - 2xy'') & -4xy''' \end{pmatrix} \begin{pmatrix} I_x \\ I_y \\ I_{y'} \\ I_{y''} \\ I_{y'''} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Reducing the system results in the following 3 equations which determine the differential invariants:

$$I_x = 0 \tag{6}$$

$$yI_y - y''I_{y''} - 2y'''I_{y'''} = 0 \tag{7}$$

$$yI_{y'} + y'I_{y''} = 0 \tag{8}$$

Using the method of characteristics, from equation (8) we have $\frac{dy'}{y} = \frac{dy''}{y'}$, *which is solved by* $c = 2yy'' - y'^2$, *giving the first fundamental differential invariant,* $r(x,y) = 2yy'' - y'^2$ *and the function* $I = I(x, y, 2yy'' - y'^2, y''')$. *Now from equation (7) the second fundamental differential invariant is a first integral of* $\frac{dy}{y} = \frac{dy''}{-y''} = \frac{dy'''}{-2y'''}$. *Solving* $\frac{dy}{y} = \frac{dy'''}{-2y'''}$, *we find* $y^2y''' = k$ *resulting in* $I = I(x, 2yy'' - y'^2, y^2y''')$. *Finally, since* $I_x = 0$, *any differential invariant function is of the form* $I(2yy'' - y'^2, y^2y''')$. *Thus the fundamental differential invariants are* $r_3 = 2yy'' - y'^2$ *and* $v_3 = y^2y'''$ *and we can now calculate higher order differential invariants as follows:*

$$\frac{dv_3}{dr_3} = \frac{D_x v_3}{D_x r_3} = \frac{2yy'y''' + y^{(iv)}y^2}{2y'y'' - 2y'y'' + 2yy'''} = y' + \frac{y^{(iv)}y}{2y'''}.$$

Substituting the original equation $y^{(iv)} = \frac{2}{y}(1-y')y'''$ *into this results in* $\frac{dv_3}{dr_3} = y' + 1 - y' = 1$, *thus the ODE of this example is equivalent to the equation* $\frac{dv_3}{dr_3} = 1$ *whose solution is* $v_3 = r_3 + c$, *or in terms of* x *and* y, $y''' = \frac{2yy'' - y'^2 + c}{y^2}$. *Thus, this 3rd order ODE is invariant under the three-parameter Lie group generated by* X_1, X_2 *and* X_3 *above. Additionally, any other ODE of order* $n \geq 3$ *which is invariant under this same group can be written as a function of the differential invariants* $v_3^{(n-3)} = F\left(r_3, v_3, \ldots, v_3^{(n-4)}\right)$.

5. Center Manifolds

We now turn our attention to some concepts from dynamical system theory. Our main objective is to find the center manifolds of a system. By reducing to the center manifold, which is a lower dimensional object than the whole system, we are able to investigate the dynamics of the system. In future work we will investigate the existence of center manifolds for systems arising in the work of the scientists at the Natick Soldier Center and then consider the dynamics of the systems based on the results on center manifolds.

First we define what we mean by an invariant manifold. Using $\dot{x}$ to denote the derivative of x with respect to time t, consider a system of differential equations $\dot{x} = F(x)$ with $x \in \mathbb{R}^n$. A set $S \subseteq \mathbb{R}^n$ is said to be an **invariant manifold** if for $x_0 \in S$, the solution $x(t)$, with $x(0) = x_0$ remains in S for all t, that is, the solution curve lies entirely in S. The manifold S is **locally invariant** if $x(t) \in S$ for $|t| < T$ for some fixed real number $T > 0$.

Given a nonlinear system of differential equations $\dot{x} = F(x)$ with a critical point at the origin, we can write the system in standard form as follows:

$$\begin{aligned} \dot{x} &= Ax + f(x,y,z) \\ \dot{y} &= By + g(x,y,z) \\ \dot{z} &= Cz + j(x,y,z) \end{aligned} \tag{9}$$

where A has eigenvalues with zero real parts, B has eigenvalues with negative real parts and C has eigenvalues with positive real parts and the functions $f(x,y,z)$, $g(x,y,z)$ and $j(x,y,z)$ are all C^2. The system gives rise to three invariant manifolds: stable, unstable and center manifolds. The stable and unstable manifolds are each unique and correspond respectively to the eigenspaces of the eigenvalues with negative and positive real parts. Thus the stable (unstable) manifold consists of all the solutions which decay toward (grow from) the origin exponentially. The eigenspace of the eigenvalues with zero real part relates to the center manifold. The center manifold is an invariant manifold $y = h(x)$ with h smooth and $h(0) = h'(0) = 0$. Geometrically we can think of the center manifold as an invariant manifold tangent to the center eigenspace at the origin. The center eigenspace is the space spanned by a basis of eigenvectors corresponding to the eigenvalues of the linearized system with zero real part. Additionally, the center manifold consists of all solutions which do not exponentially decay to or expand from the origin.

According to Guckenheimer and Holmes [GH] we can assume that the unstable manifold is empty for the sake of simplicity and because physically this is most interesting. Thus we consider the system:

$$\begin{aligned} \dot{u} &= \tilde{A}u + \tilde{f}(u,v) \\ \dot{v} &= \tilde{B}v + \tilde{g}(u,v) \end{aligned} \tag{10}$$

Given a system of this form (10), with $u \in \mathbb{R}^n,\ v \in \mathbb{R}^m$, if it is not in standard form, we seek to write it as such. We first find the eigenvalues of the linearized system. The corresponding eigenvectors give a transformation matrix T such that $\begin{pmatrix} u \\ v \end{pmatrix} = T \begin{pmatrix} x \\ y \end{pmatrix}$. This allows us to write the system in standard form as follows:

$$\begin{aligned} \dot{x} &= Ax + f(x,y) \\ \dot{y} &= By + g(x,y) \end{aligned} \tag{11}$$

where $x \in \mathbb{R}^n,\ y \in \mathbb{R}^m$, and A and B are constant matrices such that the eigenvalues of A have zero real parts and the eigenvalues of B have negative real parts. The functions f and g are C^2 with $f(0,0) = g(0,0) = 0$ and $f'(0,0) = g'(0,0) = 0$, where $f'(0,0) = Df(0)$, the Jacobian matrix at zero. If we can write a given system in the above form we are guaranteed the existence of center manifolds by a standard result of center manifold theory. That is, if we can write our system so that the linear terms have at least one eigenvalue with zero real part and one with negative real part, we have center manifolds. See [GH] or [Ca] for the proof of this result.

To find a center manifold, we must solve the following system:

$$\begin{aligned} h(0) &= 0 \\ h'(0) &= 0 \\ h'(x)\,(Ax + f(x, h(x))) &= Bh(x) + g(x, h(x)) \end{aligned} \tag{12}$$

which follows from $y = h(x)$ and $\dot{h}(x) = h'(x)\dot{x}$. Solving this system is as difficult as solving the original system, so we must determine a way to approximate the solution to any desired degree of accuracy. For a function $\phi : \mathbb{R}^n \to \mathbb{R}^m$, which is C^1 in a neighborhood of the origin, define

$$(M\phi)(x) = \phi'(x)\,(Ax + f(x, \phi(x))) - B\phi(x) - g(x, \phi(x)).$$

Note that $(Mh)(x) = 0$. The following theorem gives that the center manifold $h(x)$ is approximated by $\phi(x)$ to the desired degree.

Theorem 5.1 ([Ca] Theorem 3 p 5)**.** *Let ϕ be a C^1 mapping of a neighborhood of the origin in $\mathbb{R}^n$ into $\mathbb{R}^m$ with $\phi(0) = 0$ and $\phi'(0) = 0$. Suppose that as x approaches 0, $(M\phi)(x) = O(|x|^q)$ where $q > 1$. Then as x approaches 0, $|h(x) - \phi(x)| = O(|x|^q)$.*

We describe how to use this result to find the center manifold. Start with a set of differential equations of the form (11). Then let $\phi(x) = ax^2 + bx^3 + \cdots$ since this is the smallest $\phi(x)$ that satisfies Theorem 5.1. Use the given system (11) to determine $(M\phi)(x)$. Define $\phi(x)$ to be the map such that $(M\phi)(x) = O(|x|^k)$ for some $k > 2$ and let $h(x) = \phi(x) + O(|x|^k)$, resulting in a center manifold $h(x)$ to a degree of accuracy of $O(|x|^k)$. If we desire a higher degree of accuracy, let $\tilde{\phi}(x) = \phi(x) + \psi(x)$ where $\psi(x) = O(|x|^k)$ and proceed with $(M\tilde{\phi})(x)$ as before. This process will be made clearer with an example.

Example 5.2 ([Ca] Example 1 p 5)**.** *Consider the system*

$$\begin{aligned} \dot{x} &= xy + ax^3 + by^2x \\ \dot{y} &= -y + cx^2 + dx^2y \end{aligned} \tag{13}$$

This system is already in standard form as $f(x, y) = xy + ax^3 + by^2x$ and $g(x, y) = cx^2 + dx^2y$, so it follows that $f(0, 0) = g(0, 0) = 0$ and $f'(0, 0) = g'(0, 0) = 0$. Additionally, in this example $A = 0$ and $B = -1$. We calculate $(M\phi)(x)$:

$$(M\phi)(x) = \phi'(x)(x\phi(x) + ax^3 + b\phi(x)^2x) + \phi(x) - cx^2 - dx^2\phi(x).$$

Let us assume that $\phi(x) = O(x^2)$. Then $\phi(x) = \alpha x^2 + \gamma x^3 + \cdots$ and we show that $(M\phi)(x) = \phi(x) - cx^2 + O(x^4)$ as follows: $\phi'(x) = 2\alpha x + 3\gamma x^2 + \cdots$, so multiplying this by $f(x, \phi(x)) = x\phi(x) + ax^3 + b\phi(x)^2$ results in $O(x^4)$ terms only. Then $(M\phi)(x) = O(x^4) + \phi(x) - cx^2 - dx^2\phi(x) = \phi(x) - cx^2 + O(x^4)$. If we define $\phi(x) = cx^2$, then $(M\phi)(x) = O(x^4)$ and $h(x) = \phi(x) + O(x^4) = cx^2 + O(x^4)$ by Theorem 5.1.

Sometimes the first approximation found is not sufficient in terms of giving enough information when considering the stability of solutions. In the above example, it is true that if $a + c = 0$, we are unable to determine the stability of the zero solution. See Carr ([Ca] p 6) for details. To obtain a better approximation for the center manifold in the system above, we let $\phi(x) = cx^2 + \psi(x)$ where $\psi(x) = O(x^4)$. Then $\phi'(x) = 2cx + \psi'(x) = 2cx + O(x^3)$ and we recalculate $(M\phi)(x)$.

$$\begin{aligned}(M\phi)(x) &= \phi'(x)(x\phi(x) + ax^3 + b\phi(x)^2x) + \phi(x) - cx^2 - dx^2\phi(x)\\ &= \phi'(x)(cx^3 + x\psi(x) + ax^3 + b\phi(x)^2x) + cx^2 + \psi(x) - cx^2 - cdx^4 - dx^2\psi(x)\\ &= -cdx^4 + \psi(x) + O(x^6)\end{aligned}$$

Therefore letting $\phi(x) = cdx^4 + cx^2$, we have $(M\phi)(x) = O(x^6)$, resulting in the center manifold $h(x) = cx^2 + cdx^4 + O(x^6)$.

It should be reiterated that a system does not admit a unique center manifold. Both $h_1(x)$ and $h_2(x)$ can be center manifolds for the same system of differential equations. In this case the difference $h_1(x) - h_2(x) = O(|x|^q)$ as $x \to 0$ for any $q > 1$. Additional properties of center manifolds can be found in Carr ([Ca] p. 28).

6. Lie Symmetries acting on Center Manifolds

Some work has been done in recent years linking the ideas of Lie symmetries and dynamical systems theory. We have already discussed that Lie symmetries leave invariant the space of solutions curves and sometimes particular solution curves within the space. We would like to find other spaces that are left invariant by the action of the symmetries. The spaces that we are interested in are those arising in dynamical systems theory, in particular, the stable, unstable and center manifolds as discussed above. The following results with proof are found in a 1994 paper of Cicogna and Gaeta, [CG].

Proposition 6.1. *(Proposition 2.4 of* [CG]*) All Lie point symmetries leave invariant both the stable and unstable manifolds of the dynamical system. Also, Lie point symmetries transform any center manifold into a center manifold (possibly the same).*

Proposition 6.2. *(Proposition 7.1 of* [CG]*) Given any center manifold of a dynamical system, there is, generically, some nontrivial Lie point symmetry leaving invariant this center manifold, and conversely any Lie point symmetry of the dynamical system leaves invariant some center manifold.*

To determine if a Lie symmetry leaves invariant a center manifold, it is necessary to have a characterization analogous to that for invariant solutions above. Recall that if $Q(x, y, y') = \eta - y'\xi = 0$, y' is a solution invariant under the symmetry given by $X = \xi\partial_x + \eta\partial_y$. Here using the notation of [CG] we let $u \equiv (x, y) \in \mathbb{R}^n$ where $x \in \mathbb{R}^k$ span the eigenspace, and $y \in \mathbb{R}^{n-k}$ the orthogonal subspace. In this case, we have a center manifold $y = h(x)$ described by $(n - k)$ equations. A Lie symmetry of the

original system of differential equations is given by $X = \phi(x,y)\partial_x + \psi(x,y)\partial_y$. We have the following result.

Proposition 6.3. [CG] *Proposition 3.1 The center manifold $y = h(x)$ remains invariant under the Lie symmetry $X = \phi(x,y)\partial_x + \psi(x,y)\partial_y$ if and only if*

$$\psi(x, h(x)) = (\partial_x h(x)) \cdot \phi(x, h(x)).$$

Example 6.4. [CG] *(7.6) Consider the dynamical system given by*

$$\begin{aligned} \dot{x_1} &= -x_1 r^2 \\ \dot{x_2} &= -x_2 r^2 \\ \dot{y} &= -y \end{aligned}$$

where $r^2 = x_1^2 + x_2^2$. One Lie symmetry of this system is given by the generator $X = x_2\partial_{x_1} - x_1\partial_{x_2}$. We will show that the center manifold $y = h(x) = F\left(\frac{x_2}{x_1}\right)\mathrm{e}^{\frac{-1}{2r^2}}$, where F is a constant, is invariant under this symmetry according to Proposition 6.3. In this example, $\psi(x, h(x)) = 0$, so we must show that $(\partial_x h(x)) \cdot \phi(x, h(x)) = 0$.

$$(\partial_x h(x)) \cdot \phi(x, h(x)) = \left\langle \frac{-x_1}{r^4}\mathrm{e}^{\frac{-1}{2r^2}}, \frac{-x_2}{r^4}\mathrm{e}^{\frac{-1}{2r^2}} \right\rangle \cdot \langle x_2, -x_1 \rangle = 0$$

Thus the Lie symmetry given by X maps this center manifold to itself.

7. Future Work

We have made progress in the understanding of the fundamental concepts of Lie symmetries and how they behave on differential equations. The next step is to continue to pursue the link between Lie symmetries and center manifolds. Lie symmetries and center manifolds each provide a way to reduce a given system to one of lesser dimension, possibly giving a more easily studied and solved equivalent system. We hope to consider both symmetries and center manifolds of systems arising in the work of scientists at the Natick Soldier Center. In most cases, the researchers have used numerical methods to solve these systems and we would like to compare their solutions to ones found through the use of symmetries and center manifolds. Additionally, we hope to be able to more fully discuss the behavior of the systems through our studies.

The first system to be considered is a quasi-chemical kinetics model used to study the growth and death of *Staphylococcus aureus* in bread [RTDFK]. Scientists at the Natick Soldier Center use this system to predict appropriate shelf-life for sandwiches, in particular considering the growth of bacteria in the region of the bread meeting the filling. The system consists of four ODEs which describe the concentration of lag phase cells (M), growth phase cells (M^*), antagonist (A), and dead cells (D). The nonnegative constants k_1 through k_4 are the concentrations and rate constants. The constant G is

the net natural growth rate and is defined as $G = k_2 - k_4$. At $t = 0$, we have the following initial conditions: $M = I$, where $I \approx 10^3 - 10^4$ and $M^* = A = D = 0$. The system follows.

$$\begin{aligned}
\dot{M} &= -k_1 M \\
\dot{M}^* &= k_1 M + M^*(G - \varepsilon A) \\
\dot{A} &= M^*(k_2 - \varepsilon A) \\
\dot{D} &= M^*(k_4 + \varepsilon A)
\end{aligned}$$

We can see that this system satisfies the necessary conditions to have a center manifold if we rewrite separating the linear and nonlinear terms as follows:

$$\begin{pmatrix} \dot{M} \\ \dot{M}^* \\ \dot{A} \\ \dot{D} \end{pmatrix} = \begin{pmatrix} -k_1 & 0 & 0 & 0 \\ k_1 & G & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & k_4 & 0 & 0 \end{pmatrix} \begin{pmatrix} M \\ M^* \\ A \\ D \end{pmatrix} + \begin{pmatrix} 0 \\ -\varepsilon M^* A \\ -\varepsilon M^* A \\ \varepsilon M^* A \end{pmatrix}.$$

The eigenvalues of the linearized system are $-k_1, G, 0$ and 0. Since k_1 is nonnegative, there is at least a dimension one stable manifold. Additionally, having zero-valued eigenvalues and the fact that the nonlinear part of the system of differential equations is zero and has zero derivative matrix at zero, guarantees the existence of a center manifold.

The calculation of a center manifold in this case is more tedious than the examples included in this paper. In future work we will find a center manifold and then also the Lie symmetries of the system in order to investigate the behavior of the system.

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