Analytical Shock Response of a Transversely Point-Loaded Linear Rectangular Plate

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1 Executive Summary

Ballistic shock, the strong vibrations that propagate through a vehicle structure after a ballistic impact, can damage electronic components that are mounted on various interior locations of the vehicle. To protect onboard critical electronic components from ballistic shock damage, predicted shock response spectra (SRS) are used as design guidelines in the selection of shock isolation devices. Current technology to predict shock response intensity at component mounting locations includes Finite Element Analysis, Statistical Energy Analysis, and various other modeling approaches. The Army Research Laboratory is implementing a process to determine component damage probability based on a numerical modal superposition method. This process allows detailed modeling of an entire vehicle (including instrumentation racks) but requires months of effort to perform skillful FEA modeling. Such an expenditure is acceptable only for selected high-value vehicles, and is impractical for analysis of many vehicles. Therefore it is desirable to investigate alternative shock analysis methods that are simple and general (i.e., independent of specific vehicle configurations). Although such methods could entail reduced accuracy, for some situations they would eliminate the need for time-consuming finite element modeling, and could provide analyses useful with a variety of vehicles. As such, the methods could be used in "screening" for rapid identification of potential shock-damage issues. If necessary, further detailed analysis could be accomplished subsequently.

When an armored vehicle is subjected to a non-perforating large-caliber munitions impact or blast, direct-mounted components that are near the impact location will always risk the highest damage potential. Live-fire ballistic tests and laboratory experiments have proved this observation, as has numerical modeling. The research presented in this paper develops, from an exact continuous model, a constant- and lumped-parameter linear matrix model of a homogeneous, rectangular, simply-supported plate subject to transverse ballistic shock. The model is provided in a form suitable for eigen-analysis intended to yield kinematic responses for SRS determination. Damping can be included in either Rayleigh or modal form. The enabled "same-plate" shock analysis, while not encompassing an entire military vehicle, would treat the most severe cases of shock response, for components mounted directly on the impacted plate.

2 Introduction

Military equipment must be designed to operate under a variety of hostile environmental conditions. Mechanical shocks, from blast loads or projectile impacts to military platforms, often place debilitating demands on critical electronic and other military subsystems. These failures can in turn disable the supported military platforms even when the impacts themselves are humanly survivable.

Various types of shock response spectra (SRSs) [1, 2] are commonly used as analytical tools for the design and analysis of military hardware that will likely be subject to remote impulsive shock loading. The remote impulse is filtered through the eigenstructure of the supported military platform to the attachment point of the supporting subsystem, which is in turn excited via its composite modes. A single point on a SRS curve graphically represents the maximum time-response of a hypothetical single-degree-of-freedom (SDOF) mass-spring-damping (MSD) system, assumed to be attached in some desired direction at a contemplated or actual subsystem attachment point. A complete SRS comprises a curve of the selected maximum response (typically absolute acceleration, relative position, or pseudovelocity [3, 4]) plotted as a function of SDOF-system frequency, for a single damping ratio. Figure 1 represents a shock-loaded system *S* with a remote impact at point *C* and an attached SDOF system at point *D*. A single point (ω_n, S_V) of an undamped pseudovelocity shock response spectrum (PSRS), for example, would represent the maximum relative position $x - d$ of mass *m*, over time, multiplied by undamped natural frequency $\sqrt{k/m}$, plotted versus that frequency:

$$S_V(\omega_n) := \omega_n \left| x(t) - d(t) \right|_{\max}. \tag{1}$$

For definitions of other common types of SRS, refer to [4].

The approach employed by most analysts, for determining the respective responses at a set of *r* attachment points D_i to an input at *C*, is to use direct integration via a finite element analysis model. For each impact point of interest on supported system *S*, the integration process must be repeated. This is a very time-consuming process. For a

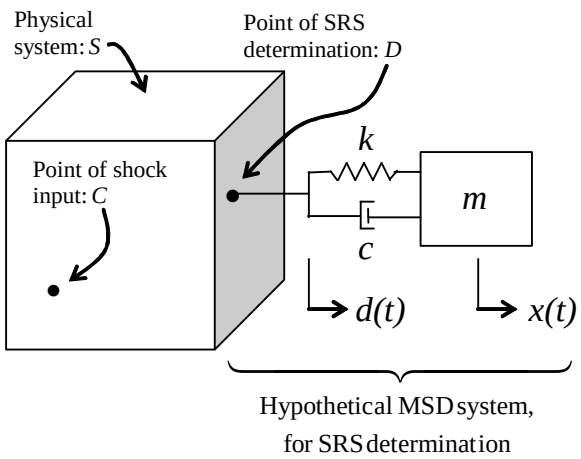

Figure 1. Shock-loaded system *S*, with (hypothetical) attached MSD system for SRS determination

typical Stryker-vehicle analysis using, say, 1E5 shell elements and 1500 impact points, this approach takes about two days per impact for a set of attachment points. Although it is too slow for many impacts, the method can be used for comparison in a few cases.

If the region of plastic deformation due to the impact is small, so that modal superposition can be used, the process is significantly faster. For a Stryker vehicle analysis using 1E5 shell elements and 500 modes, the modal analysis time is about 72 hours. The subsequent transient analysis time for a set of attachment points then takes about 10 minutes, for each impact location. Though much faster than the direct integration approach, with 1500 impact points the transient analysis time is still about 10 days. Then the response of the attached SDOF system must be evaluated numerically via a convolution integral (or some approximation of one) for each desired combination of impulse shape, impact location, attachment point, attached-system damping ratio, and attached-system natural frequency. The results from either approach are typically compiled in a huge database for use in subsystem design and analysis. Due to the immense computational overhead required with either of the above processes, methods are being sought which can speed up SRS development for a supported military platform.

As noted above, modal analysis offers the faster of the two routes to attachment-point transient response, provided the system model can be assumed to be linear (as is frequently realistic for point impacts). If the remote shock input (e.g., at *C*, Fig. 1) can be idealized as a Dirac-delta function, then for Rayleigh or modal damping with symmetric mass and stiffness matrices the attachment-point transient response (e.g., at point *D*, Fig. 1) can be determined analytically from the basic platform's modal structure [5, 6]. In these cases, the system response is found using 2nd-order linear matrix differential equations. For more general damping or with an asymmetric mass or stiffness matrix a 1st-order linear matrix differential equation (i.e., state-space form) can be used [7]. In any of the above cases the response will be a linear combination of exponentially decaying sinusoids of known parameters. With Rayleigh damping the response has also

been determined analytically for other idealizations of the remote input—such as a rectangular pulse or a sawtooth pulse [6]. The existence of an analytical solution, as in any of the foregoing cases, means that the appropriate structure (modal or eigen-) of the basic platform need be determined only once, irrespective of the number of impacts or attachment points. This in turn offers an enormous savings in modeling and computational time and effort.

Further, from these analytical solutions the SDOF system's kinematic response can also be determined analytically [5, 6, 7]. Consequently, each convolution integral that would otherwise be required to determine an SRS (i.e., for each undamped natural frequency and damping ratio, at each attachment point), can be replaced with an algebraic equation. The computational savings is not significant for any single data point on an individual SRS plot; but since the number of convolutions that must be performed to develop the platform's data base can be several hundred thousand, the additional computational savings can be quite substantial. Additionally, use of algebraic kinematic responses, for attachment point and for SDOF-system mass, means that SRS data points can be determined exactly for arbitrarily low frequencies. This would not be possible if nonalgebraic means were used for evaluating the convolution integrals [3].

The above paragraphs address the computational savings from impact idealization, system linearization, modal- or eigen-decomposition, and a priori (algebraic) convolution. It is also possible, with some platforms, to represent an impacted surface by a plate with idealized geometry, material properties, and boundary conditions. For many such cases exact continuous solutions, or continuous solution-approximations or bounds, exist in the literature [8, 9, 10]. The simplest case for which an exact solution exists is a homogeneous, undamped rectangular plate, simply supported on all edges, and subjected to a transverse external force density. The present paper develops, from this continuous solution, a linear, constant-parameter, lumped-parameter, dynamical model of a homogeneous, simply-supported rectangular plate, to include either Rayleigh or modal damping. The dynamical equations are given in a 2nd-order matrix form conducive to using the analytical idealizations and simplifications offered above for the 1st-order matrix form.

3 Problem Statement

Consider a flat, linear, homogenous rectangular plate (Fig. 2) of length a (x-direction), width b (y-direction), uniform thickness h, Young's modulus E, Poisson's ratio ν, and mass density ρ per unit area; simply supported on all four edges; and subject to transverse external force intensity (i.e., per unit area of plate surface),

$$q = q(x, y). \tag{2}$$

The x and y axes lie in the undeformed neutral plane.

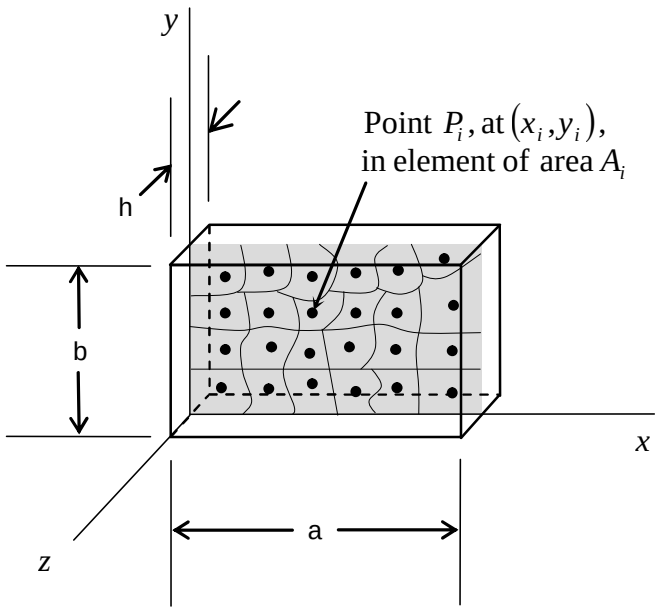

Figure 2. Simply-supported plate
to be discretized

The partial differential equation for the plate is well-known (e.g., [9, p. 257]):

$$\rho w_{tt} + D\nabla^4 w = q(x, y); \tag{3}$$

where

$$w = w(x, y, t) \tag{4}$$

is the transverse (z-direction) displacement of the neutral surface,

the flexural rigidity D is defined by

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)}, \tag{5}$$

the subscript notation indicates partial differentiation:

$$w_{tt} := \frac{\partial^2 w}{\partial t^2}, \tag{6}$$

and

$$\nabla^4 := \nabla^2 \nabla^2 \tag{7}$$

is the biharmonic differential operator (the dual application of the Laplacian operator ∇^2). For rectangular coordinates the Laplacian is expressed by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{8}$$

The simply supported boundary conditions [9] are represented by

$$w = w_{xx} + \nu w_{yy} = 0. \tag{9}$$

An analytical, modal solution to the homogeneous form of the differential equation is also well known, for this plate geometry and set of boundary conditions. In particular,

$$w(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}(x, y)\eta_{mn}(t), \tag{10}$$

where undamped natural frequency *mn* is

$$\omega_{mn} = \sqrt{\frac{D}{\rho}}\pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right], \tag{11}$$

the associated modeshape is

$$W_{mn} = B_{mn}\sin\left(\frac{m\pi\, x}{a}\right)\sin\left(\frac{n\pi\, y}{b}\right), \tag{12}$$

and the corresponding undamped modal coordinate is

$$\eta_{mn} = \sin(\omega_{mn} t + \phi_{mn}). \tag{13}$$

Constants B_{mn} and ϕ_{mn} depend on the initial conditions.

If the undamped natural frequencies are arranged in increasing order, then indices *mn* can be replaced with a single index *k*, and (10) through (13) can be rewritten as follows:

$$w(x, y, t) = \sum_{k=1}^{\infty} W_k(x, y)\eta_k(t) \tag{14}$$

where the kth undamped natural frequency is

$$\omega_k = \sqrt{\frac{D}{\rho}}\,\pi^2\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right], \tag{15}$$

the associated modeshape is

$$W_k = B_k \sin\left(\frac{m\pi\, x}{a}\right)\sin\left(\frac{n\pi\, y}{b}\right) = B_k \sin\left(\frac{m\pi\, x}{a}\right)\sin\left(y\sqrt{\omega_k\sqrt{\frac{\rho}{D}} - \left(\frac{m\pi}{a}\right)}\right), \tag{16}$$

and the corresponding undamped modal coordinate is

$$\eta_k = \sin(\omega_k t + \phi_k). \tag{17}$$

The goals of the present paper are as follows:

(a) to use the plate's partial differential equation (3) and its solution (14) through (17) to develop a lumped-parameter differential equation for the plate in a useful linearized matrix form;

(b) to use the matrix differential equation of motion to determine analytically the response of the plate at arbitrary point (x_j, y_j) to an impulsive shock input at a different arbitrary point (x_i, y_i); and

(c) to use the response solution at (x_j, y_j) to determine analytically the kinematic quantities needed for SRS computation at (x_j, y_j), under the assumption of various types of idealized impulsive loading at (x_i, y_i).

4 Solution

4.1 Matrix differential equation of motion

In terms of differential mass elements

$$dm = \rho\, dA, \tag{18}$$

(3) can be re-expressed, still for the continuous system, as

$$(\rho dA)\, w_{tt} + (D dA)\nabla^4 w = q(x, y)\, dA\,. \tag{19}$$

To express the mathematical model in finite-dimensional form, first discretize the plate into r elements of respective areas A_i $(i = 1,\ldots, r)$, with the mass

$$m_i = \rho A_i \tag{20}$$

of the ith element located in the element on the neutral surface, at some point P_i having coordinates (x_i, y_i). (Refer to Fig. 2.) The fidelity of the discretized representation can be expected to vary, of course, depending on the choices both of the value of r and of the locations of points P_i; a reasonable location for P_i would be at the centroid of the ith mass element. Lump the transverse loading on the ith element into concentrated force

$$f_i(t) = \int_{A_i} q(x, y) dA_i, \tag{21}$$

applied at P_i. Then for the ith element (19) becomes

$$m_i \, w_{tt}\big|_{(x_i, y_i)} + (D A_i) \nabla^4 w\big|_{(x_i, y_i)} = f_i. \tag{22}$$

In vector form, using transparent notation for the vector elements,

$$\underbrace{\begin{bmatrix} m_1 & & \\ & \ddots & \\ & & m_r \end{bmatrix}}_{M} \underbrace{\begin{Bmatrix} \ddot{w}_1 \\ \\ \ddot{w}_r \end{Bmatrix}}_{\underline{\ddot{w}}} + D \underbrace{\begin{Bmatrix} A_1 \nabla^4 w_1 \\ \vdots \\ A_r \nabla^4 w_r \end{Bmatrix}}_{DA\nabla^4 \underline{w}} = \underbrace{\begin{Bmatrix} f_1 \\ \vdots \\ f_r \end{Bmatrix}}_{\underline{f}}, \tag{23}$$

where

$$A = \begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_r \end{bmatrix} \tag{24}$$

and

$$M = \rho A. \tag{25}$$

Application of the biharmonic operator to (14) leads directly to

$$\nabla^4 w = \nabla^4 \left[\sum_{k=1}^{\infty} W_k(x, y) \eta_k(t) \right] = \sum_{k=1}^{\infty} \left\{ \nabla^4 [W_k(x, y)] \eta_k(t) \right\}. \tag{26}$$

Expanding ∇^4 in terms of rectangular coordinates via (8), applying it to (26), and approximating the modal sum by the first r terms, one obtains

$$\nabla^4 w = \frac{\rho}{D}\sum_{k=1}^{r}\omega_k^2 W_k(x,y)\eta_k(t), \tag{27}$$

where each continuous modeshape W_k is defined by (16) (repeated below):

$$W_k(x,y) = B_k \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{28}$$

or, equivalently,

$$W_k(x,y) = B_k \sin\left(\frac{m\pi x}{a}\right)\sin\left(y\sqrt{\omega_k\sqrt{\frac{\rho}{D}} - \frac{m\pi}{a}}\right). \tag{29}$$

The modeshapes can be discretized into vector form:

$$\underline{u}_k = \begin{Bmatrix} B_k \sin\left(\frac{m\pi x_1}{a}\right)\sin\left(\frac{n\pi y_1}{b}\right) \\ \vdots \\ B_k \sin\left(\frac{m\pi x_r}{a}\right)\sin\left(\frac{n\pi y_r}{b}\right) \end{Bmatrix}, \tag{30}$$

which can be normalized using the Euclidean norm:

$$\underline{m}_k = \frac{\underline{u}_k}{|\underline{u}_k|}. \tag{31}$$

Define $r \times p$ modeshape matrix

$$U = \left[\underline{m}_1, ..., \underline{m}_p\right], \tag{32}$$

and use it to define a normalized modal coordinate vector $\underline{\eta}$ as follows:

$$\underline{w} = U\underline{\eta}. \tag{33}$$

It should be noted here that the number p of modes comprising U can never exceed half the number r of mass elements—and in many cases p must be considerably smaller—in order to avoid aliasing effects. In particular, for a rectangular plate comprising r equal rectangular elements of dimensions $\Delta x \times \Delta y$, discretized modeshape $\underline{u}_k$ cannot represent continuous modeshape U_k without aliasing unless

$$\Delta x < a / m \tag{34}$$

and $$\Delta y < b / n \tag{35}$$

for the particular values of m and n corresponding to U_k.

Now, using (33), (25), and (27), the discretized equation of motion (23) can be expressed in terms of modal coordinates by

$$MU\ddot{\underline{\eta}} + MU\Omega^2 \underline{\eta} = \underline{f}\,, \tag{36}$$

for $$\Omega = \begin{bmatrix} \omega_1 & & \\ & \ddots & \\ & & \omega_p \end{bmatrix}. \tag{37}$$

In terms of physical coordinates, and using U^+ to represent the pseudoinverse of U,

$$M\ddot{\underline{w}} + K\underline{w} = \underline{f}\,, \tag{38}$$

where $$K = MU\Omega^2 U^+. \tag{39}$$

The pseudoinverse U^+ was required above in lieu of the normal inverse, since U is not square.

It should be noted here, for subsequent use, that although mass matrix M is symmetric—and diagonal (23)—stiffness matrix K is not necessarily even symmetric, as one can show readily by using $r = 2$ and solving for K algebraically. Further, the discretized modeshape matrix U is not necessarily unitary.

One can now insert modal damping into (36). First premultiply each term of (36) by $(MU)^+$ to obtain

$$\ddot{\underline{\eta}} + \Omega^2 \underline{\eta} = (MU)^+ \underline{f}\,. \tag{40}$$

Define a modal damping matrix by

$$C_M = \begin{bmatrix} 2\varsigma_1\omega_1 & & \\ & \ddots & \\ & & 2\varsigma_p\omega_p \end{bmatrix}. \tag{41}$$

Then from (40) the damped equations are

$$\ddot{\underline{\eta}} + C_M \dot{\underline{\eta}} + \Omega^2 \underline{\eta} = (MU)^+ \underline{f} . \tag{42}$$

Note that these equations are decoupled. In terms of physical coordinates, via (33), (42) reduces to

$$M \ddot{\underline{w}} + C \dot{\underline{w}} + K \underline{w} = \underline{f} , \tag{43}$$

where

$$C = MUC_M U^+ , \tag{44}$$

and K is as defined previously, by (39).

Alternatively, with Rayleigh damping, (36) becomes

$$M \ddot{\underline{w}} + C_R \dot{\underline{w}} + K \underline{w} = \underline{f} , \tag{45}$$

where

$$C_R = \alpha M + \beta K , \tag{46}$$

for some desired α, β. In terms of modal coordinates, (45) is

$$MU\ddot{\underline{\eta}} + C_R U \dot{\underline{\eta}} + MU\Omega^2 \underline{\eta} = \underline{f} , \tag{47}$$

or

$$\ddot{\underline{\eta}} + (MU)^+ C_R U \dot{\underline{\eta}} + \Omega^2 \underline{\eta} = (MU)^+ \underline{f} . \tag{48}$$

This reduces in turn to the decoupled set of equations

$$\ddot{\underline{\eta}} + \Gamma \dot{\underline{\eta}} + \Omega^2 \underline{\eta} = (MU)^+ \underline{f} , \tag{49}$$

where, using $r \times r$ identity matrix I,

$$\Gamma = \alpha I + \beta \Omega^2 = \begin{bmatrix} \alpha + \beta \omega_1^2 & & \\ & \ddots & \\ & & \alpha + \beta \omega_p^2 \end{bmatrix} . \tag{50}$$

Alternatively expressed,

$$\Gamma = \alpha I + \beta \Omega^2 = \begin{bmatrix} 2\varsigma_1 \omega_1 & & \\ & \ddots & \\ & & 2\varsigma_p \omega_p \end{bmatrix}, \tag{51}$$

where

$$\varsigma_i = \frac{\alpha + \beta \omega_i^2}{2\omega_i}. \tag{52}$$

Observe that since K is not symmetric, and since U is not unitary, the usual unitary similarity transformation $U^*(\ )U$ has not been (cannot be) used to diagonalize the damping or stiffness matrix in either (43) or (45). However, (42) and (49) still represent the respective modal equations in decoupled form.

4.2 Response $w(x_j, y_j, t)$ to shock input $f(x_i, y_i, t)$

Using Laplace transforms, and assuming all modes to be underdamped, the ith modal solution to either (42) or (49) is

$$\eta_i(t) = e^{-\varsigma_i \omega_{ni} t}\left(A_i \cos\omega_{di} t + B_i \sin\omega_{di} t\right) + \frac{1}{\omega_{di}}\left(e^{-\varsigma_i \omega_{ni} t} \sin\omega_{di} t\right) * \tilde{f}_i\ , \tag{53}$$

where

$$A_i = \eta_i\left(0^+\right), \tag{54}$$

$$B_i = \frac{\varsigma_i \omega_{ni}}{\omega_{di}} \eta_i\left(0^+\right) + \frac{1}{\omega_{di}} \dot{\eta}_i\left(0^+\right), \tag{55}$$

and $\tilde{f}_i$ is the ith element of $(MU)^+ \underline{f}$.

In matrix form, the modal coordinate vector is

$$\underline{\eta}(t)=\begin{bmatrix} e^{-\varsigma_1\omega_{n1}t}\left[\cos\omega_{d1}t+\left(\frac{\varsigma_1\omega_{n1}}{\omega_{d1}}\right)\sin\omega_{d1}t\right] & & \\ & \ddots & \\ & & e^{-\varsigma_p\omega_{np}t}\left[\cos\omega_{dp}t+\left(\frac{\varsigma_p\omega_{np}}{\omega_{dp}}\right)\sin\omega_{dp}t\right] \end{bmatrix}\underline{\eta}(0+)$$

$$+\begin{bmatrix} \left(\frac{1}{\omega_{d1}}\right)e^{-\varsigma_1\omega_{n1}t}\sin\omega_{d1}t & & \\ & \ddots & \\ & & \left(\frac{1}{\omega_{dp}}\right)e^{-\varsigma_p\omega_{np}t}\sin\omega_{dp}t \end{bmatrix}\dot{\underline{\eta}}(0+)$$

$$\left.+\begin{bmatrix} \left(\frac{1}{\omega_{d1}}\right)e^{-\varsigma_1\omega_{n1}t}\sin\omega_{d1}t & & \\ & \ddots & \\ & & \left(\frac{1}{\omega_{dp}}\right)e^{-\varsigma_p\omega_{np}t}\sin\omega_{dp}t \end{bmatrix}*(MU)^+\underline{f}(t)\right\}. \qquad (56)$$

Accordingly, upon applying (33), the physical coordinate vector is

$$\underline{w}(t)=U\left\{\begin{bmatrix} e^{-\varsigma_1\omega_{n1}t}\left[\cos\omega_{d1}t+\left(\frac{\varsigma_1\omega_{n1}}{\omega_{d1}}\right)\sin\omega_{d1}t\right] & & \\ & \ddots & \\ & & e^{-\varsigma_p\omega_{np}t}\left[\cos\omega_{dp}t+\left(\frac{\varsigma_p\omega_{np}}{\omega_{dp}}\right)\sin\omega_{dp}t\right] \end{bmatrix}U^+\underline{w}(0+)\right.$$

$$+\begin{bmatrix} \left(\frac{1}{\omega_{d1}}\right)e^{-\varsigma_1\omega_{n1}t}\sin\omega_{d1}t & & \\ & \ddots & \\ & & \left(\frac{1}{\omega_{dp}}\right)e^{-\varsigma_p\omega_{np}t}\sin\omega_{dp}t \end{bmatrix}U^+\dot{\underline{w}}(0+)$$

$$\left.+\begin{bmatrix} \left(\frac{1}{\omega_{d1}}\right)e^{-\varsigma_1\omega_{n1}t}\sin\omega_{d1}t & & \\ & \ddots & \\ & & \left(\frac{1}{\omega_{dp}}\right)e^{-\varsigma_p\omega_{np}t}\sin\omega_{dp}t \end{bmatrix}*(MU)^+\underline{f}(t)\right\}. \qquad (57)$$

Suppose that the only nonzero element of $\underline{f}$ is a shock input

$$f_i = f\left(x_i, y_i, t\right), \tag{58}$$

applied to point P_i. Let the (i, j) elements of U and U^+ be represented by u_{ij} and v_{ij}. Then for zero initial conditions (57) reduces readily to

$$\underline{w}(t) = \left[\underline{m}_1, .., \underline{m}_p\right] \begin{Bmatrix} \dfrac{1}{\omega_{d1}} e^{-\varsigma_1 \omega_{n1} t} \sin\omega_{d1} t * \displaystyle\sum_{i=1}^{r} \frac{v_{1i}}{m_i} f_i(t) \\ \vdots \\ \dfrac{1}{\omega_{dp}} e^{-\varsigma_p \omega_{np} t} \sin\omega_{dp} t * \displaystyle\sum_{i=1}^{r} \frac{v_{pi}}{m_i} f_i(t) \end{Bmatrix}, \tag{59}$$

from which the displacement response w_j (at point P_j) to the force input f_i (at P_i) is

$$w_j(t) = \frac{f_i(t)}{m_i} * \sum_{k=1}^{p} \left(\frac{u_{jk} v_{ki}}{\omega_{dk}} \right) e^{-\varsigma_k \omega_{nk} t} \sin\omega_{dk} t\,. \tag{60}$$

If the shock input can be idealized as a Dirac delta of strength γ, the response is simply the sum of p exponentially decaying sinusoids:

$$w_j(t) = \frac{\gamma}{m_i} \sum_{k=1}^{p} \left(\frac{u_{jk} v_{ki}}{\omega_{dk}} \right) e^{-\varsigma_k \omega_{nk} t} \sin\omega_{dk} t\,, \tag{61}$$

as expected. Adding to (61) the effects of initial conditions, from (57), is straightforward, and not included here.

For certain other deterministic inputs f_i (e.g., rectangular, sawtooth, haversine, or versed-sine impulses) the terms of (60) can also be evaluated as algebraic expressions, upon performing the convolution integrations analytically (rather than numerically). Reference [6] shows the procedure for ideal-impulse (Dirac-delta), rectangular-pulse, and sawtooth-pulse inputs, in the related case of symmetric mass and stiffness matrices with Rayleigh damping. For the case of more general damping, and with asymmetric mass and/or stiffness matrices (as here), [7] provides a general procedure using a state-space approach; and achieves an algebraic solution for a Dirac-delta shock input.

4.3 Determination of shock response spectra for selected deterministic inputs

For a Dirac-delta shock input, and with Rayleigh or modal damping, the linearized system response has been shown to be a sum of easily determined, exponentially decaying sinusoids available in algebraic form (61). For rectangular-pulse and sawtooth inputs, the response would also include step and ramp components, and some time delays [6]. From these the remaining kinematic quantities behind common forms of shock response spectra could be found analytically as well, without requiring further convolution. See [5, 6, 7] for details.

5 Verification and Matlab Implementation

Verification of the above analytical model is in progress; the verification procedure follows, with some preliminary results. For purpose of comparison, two models have been implemented of an isotropic, homogeneous, aluminum test-plate: an analytical model, in Matlab; and a linear finite-element (FEA) model, in ANSYS. The plate material is 6061-T6 aluminum, with Young's modulus 69 GPa, mass density 2700 kg/m^3,

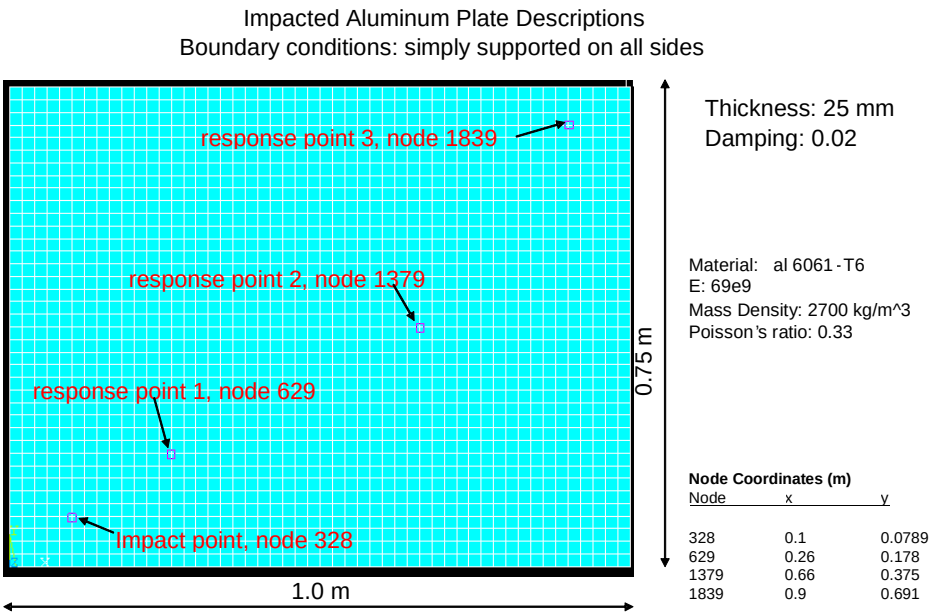

Figure 3. Test plate, showing impact- and response points

Poisson's ratio 0.33, and modal damping ratio 0.02 (all modes). (See Fig. 3.) The plate is simply supported along each of its four edges, with dimensions 1.0 m x 0.75 m x 25 mm. Both models comprise uniform rectangular (parallelepiped) elements: 50 element divisions along the length (*x*-direction, measured in the neutral plane from the lower left corner), and 38 along the width (*y*-direction, measured correspondingly). Impulses of identical strength (50 N-s) were applied at corresponding impact points, and the responses were determined at three sets of corresponding response points, from zero

initial conditions. For the plots that follow the (x, y) coordinates of the impact and response points were (0.1, 0.0789) m and (0.66, 0.375) m, respectively. A single 10-μs symmetric sawtooth pulse (Fig. 4) was used for the finite-element model; the analytical model assumed an ideal (Dirac-delta) impulse. Twenty modes were used to determine the response of each model.

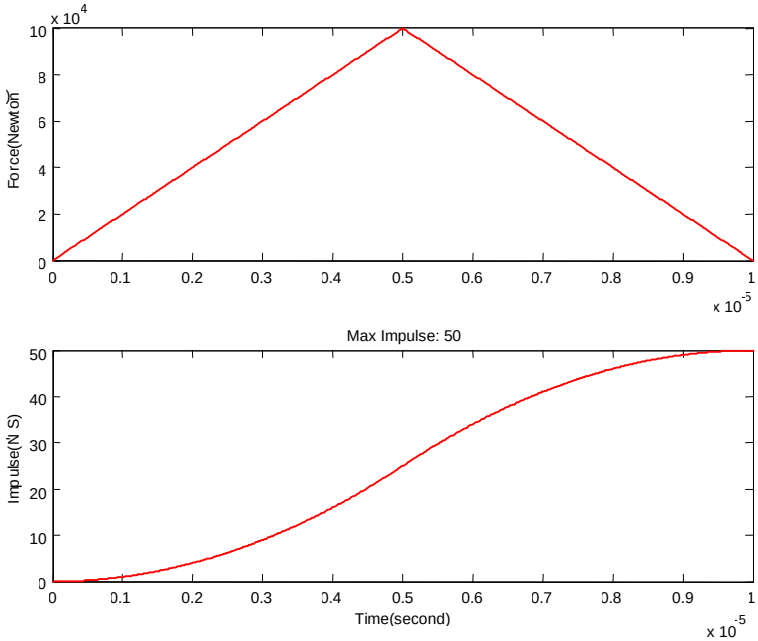

Figure 4. Loading function for FEA model

Figure 5 provides a side-by-side comparison of the 20 lowest modal frequencies (rad/s) from the respective models. In no case did the analytical frequency differ from the FEA frequency by more than 0.25 %. Figures 6 and 7 offer representative modeshape comparisons; and Figure 8 plots response-point displacements from the two models, at Node 1379 (refer to Fig. 3). The associated acceleration plots are shown in Figure 9. Corresponding modal frequencies, modeshapes, displacements, and accelerations all match well between the two models.

FEA – Ansys	Analytical	Difference (%)
168.6	168.6	0.015
350.6	350.7	0.030
492.2	492.4	0.027
654.0	654.3	0.041
674.1	674.5	0.061
977.1	978.0	0.093
1031.6	1032.0	0.042
1078.7	1079.3	0.052
1213.1	1214.2	0.085
1401.4	1403.0	0.119
1515.6	1517.7	0.137
1624.6	1625.6	0.065
1786.4	1787.5	0.061
1939.1	1942.7	0.185
1946.6	1949.4	0.142
1967.5	1969.6	0.108
2269.3	2273.2	0.173
2291.6	2293.4	0.079
2483.4	2489.0	0.228
2612.9	2617.2	0.163

Figure 5. Comparison of modal frequencies: 1st 20 modes

Mode shape comparison

(6th mode, m = 3, n = 2)

FEA - ANSYS

Analytical - Matlab

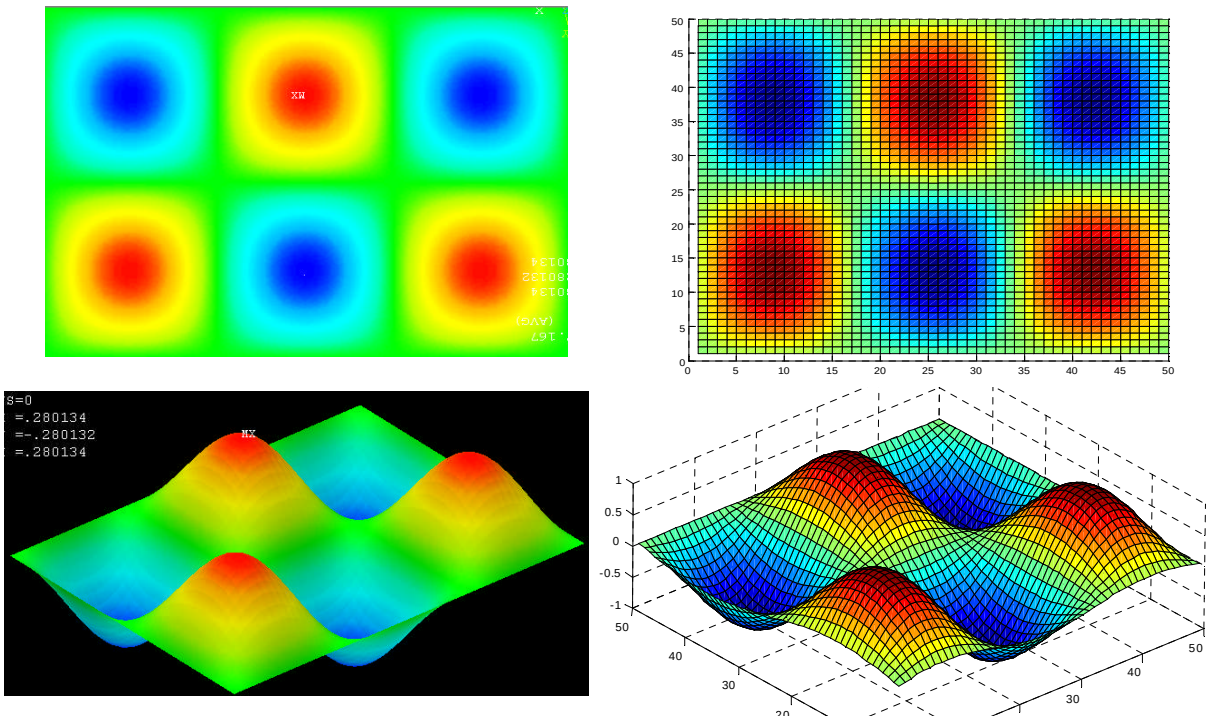

Figure 6. Modeshape comparison, 6th mode

Mode shape comparison

(19th mode, m = 5, n = 3)

FEA - ANSYS

Analytical - Matlab

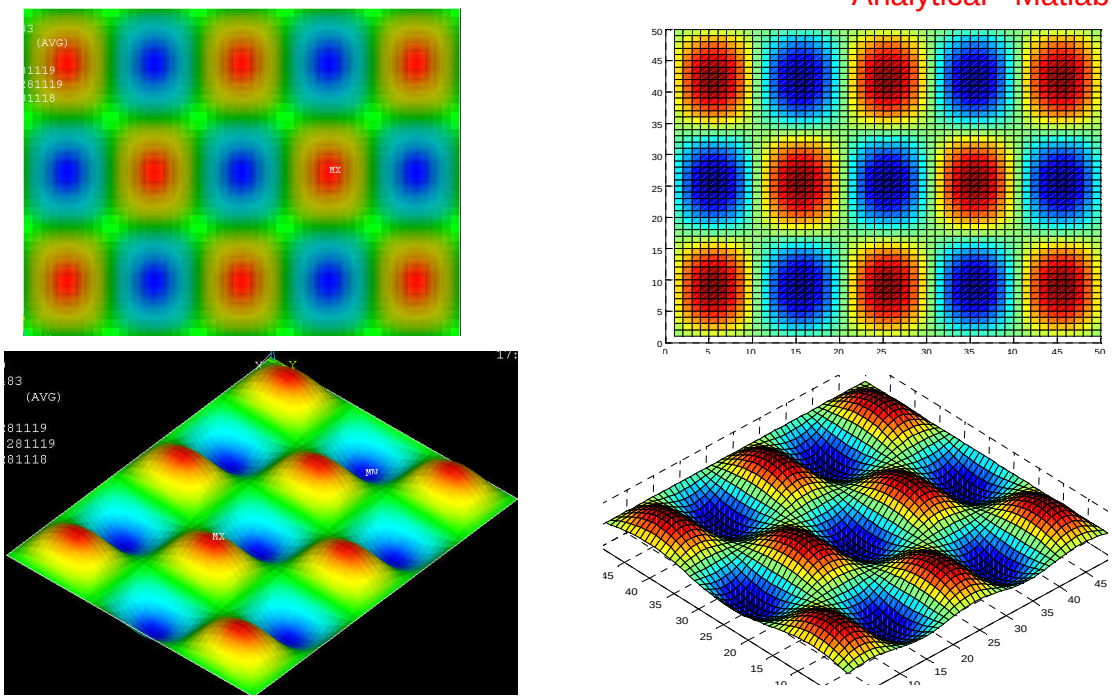

Figure 7. Modeshape comparison, 19th mode

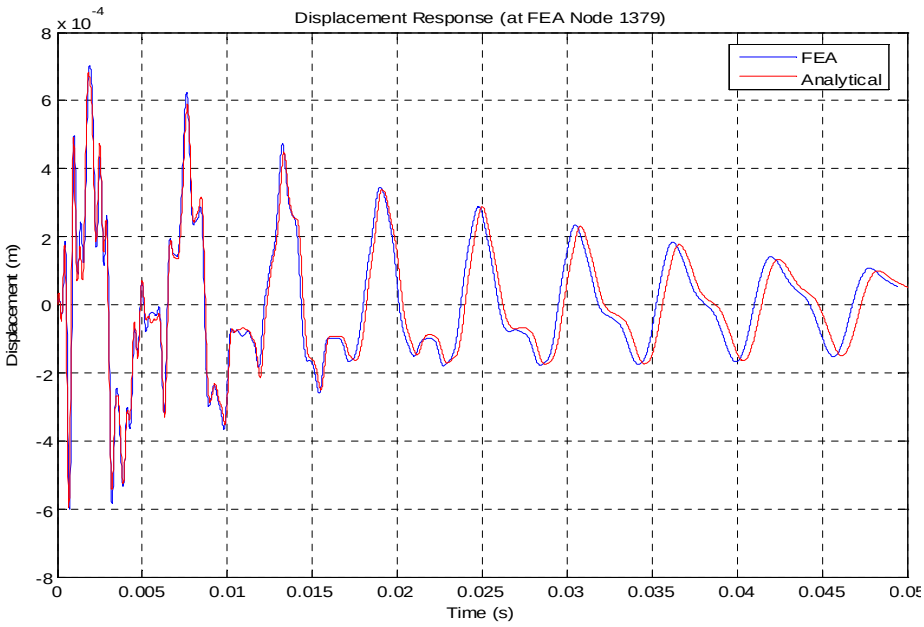

Figure 8. Response-point displacement comparison, node 1379, for an equal no. of elements (1900) and of modal components (20) in both models. Damping ratio: 0.02 for all modes.

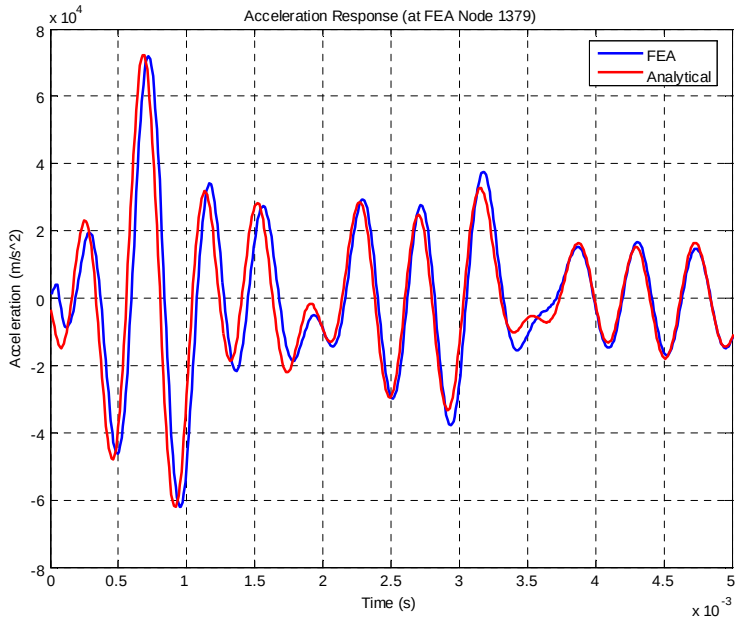

Figure 9. Response-point acceleration comparison, node 1379, for an equal no. of elements (1900) and of modal components (20) in both models. Damping ratio: 0.02 for all modes.

6 Conclusion

This paper has documented the development of a discretized (lumped-parameter) analytical model, in the form of a linear matrix differential equation of motion, for a simply supported rectangular plate subject to arbitrary transverse ballistic shock loading. Although the model was derived from a continuous (and exact) analytical model of an undamped plate, it was augmented to include either Rayleigh or modal damping. The augmented equations were then solved analytically for the case of an ideal-impulse, transverse point-shock load. A Matlab implementation was benchmarked against a linear FEA model, with comparisons made using the first 20 modes. The frequencies, modeshapes, and displacement and acceleration responses all match well.

The model is provided in a form suitable for using a *one-time* eigen-analysis of a system to yield system kinematic responses that are useful, in turn, for various SRS determinations—that is, given an *arbitrary* impact-point/response-point pair. The enabled "same-plate" shock analysis, while not encompassing an entire military vehicle, would treat the most severe cases of shock response, for components mounted directly on the impacted plate.

7 Acknowledgments

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