Optimal Design of Multi-Layered Stress Wave Attenuators

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1 Abstract

The goal of this study is to examine the efficiency and accuracy of several non-derivative based minimization algorithms for the design of stress wave attenuators. The physical model under consideration is a one-dimensional composite, composed of elastic or viscoelastic layers, subject to a suddenly applied stress. The performance of each composite is measured by the peak of the stress experienced at a point in the composite that is distant from the applied load. We seek to minimize the magnitude of this distant stress as a function of the material properties of each layer. Analytical solutions exist for a two-layered Goupillaud material, and the optimal designs from that case will be used as a test for our purely computational methods. We also base our choice of non-derivative based search algorithms upon the Goupillaud solution, which displays cusp-like behavior.

2 Introduction

Consider a composite material subject to a suddenly applied stress. As the stress wave propagates it crosses the boundaries between layers, part of the wave's energy is transmitted through the boundary and part is reflected back towards the source. Through superposition of the many reflections and transmissions a complicated wave pattern emerges that depends on the contrast between the material properties of the individual layers, the geometry of the layers, and even the order in which layers are placed, [1, 3, 15]. The goal of a stress wave attenuator is to find combinations of materials and geometries that will reduce the peak amplitude of this complicated wave phenomenon.

The computational issue of designing optimal composites is best illustrated by a simple example. Consider a composite of only two elastic layers of equal width with the material properties (elastic modulus and density) scaled by the same factor.

$$E_2 = \frac{E_1}{\alpha}$$

$$\rho_2 = \frac{\rho_1}{\alpha}$$

This guarantees that $c_1 = c_2$, the wave speed is the same in each layer. This is a special case of a problem considered by Velo and Gazonas [15], where an elastic Goupillaud composite was optimized. Our composite is illustrated in Figure 1.

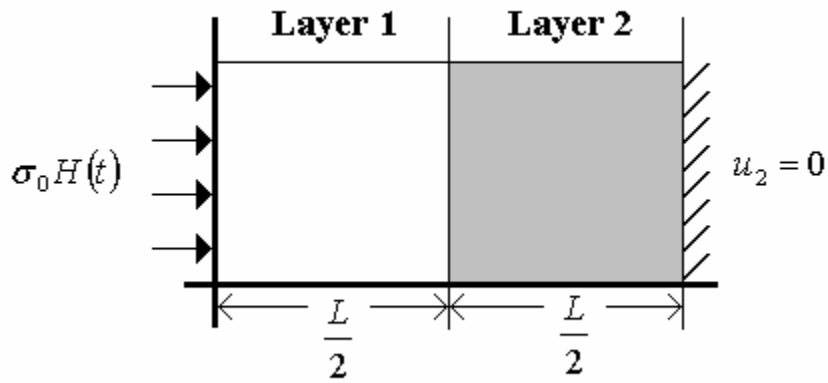

Figure 1: A two-layered composite with equal length elastic layers.

Let $u_i(x,t)$ be the displacement in layer i. The equations of motion are

$$\ddot{u}_1 = c^2 \frac{\partial^2 u_1}{\partial x^2} \text{ and } \ddot{u}_2 = c^2 \frac{\partial^2 u_2}{\partial x^2}.$$

A step is stress is applied at $x = 0$ and the composite is fixed at $x = L$, the corresponding boundary conditions are

$$\sigma_0 H(t) = E_1 \frac{\partial u_1}{\partial x}(0,t) \text{ and } 0 = u_2(L,t).$$

Assuming a perfectly bonded composite, the displacement and stress will be continuous across the layer interface:

$$u_1\left(\frac{L}{2}-,t\right) = u_2\left(\frac{L}{2}+,t\right)$$
$$E_1 \frac{\partial u_1}{\partial x}\left(\frac{L}{2}-,t\right) = \frac{E_1}{\alpha}\frac{\partial u_2}{\partial x}\left(\frac{L}{2}+,t\right)$$

This problem can be solved any number of ways. Our method of choice will be to use Laplace transforms. Using a hat to denote Laplace transformed quantities

$$\hat{f}(s) = \int_0^\infty f(t) e^{-st} dt,$$

and assuming that both materials are at rest before the stress step is applied we can find the transformed solutions for the displacements in layer 2 to be

$$\hat{u}_2 = \hat{A}_2 \sinh\left(\frac{s}{c}(x-L)\right).$$

The coefficient $\hat{A}_2$ depends on the transform variable s, and can be found by solving the system

$$\begin{bmatrix} \cosh\left(\frac{sL}{2c}\right) & \sinh\left(\frac{sL}{2c}\right) \\ \frac{E_1 s}{c}\sinh\left(\frac{sL}{2c}\right) & -\frac{E_1 s}{\alpha c}\cosh\left(\frac{sL}{2c}\right) \end{bmatrix} \begin{bmatrix} \hat{A}_1 \\ \hat{A}_2 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_0 c}{E_1 s^2}\exp\left(-\frac{sL}{2c}\right) \\ -\frac{\sigma_0}{s}\exp\left(-\frac{sL}{2c}\right) \end{bmatrix}$$

Our primary concern will be to calculate the stress at $x = L$, whose transform is found easily enough from the formula for $\hat{u}_2$.

$$\begin{aligned} \hat{\sigma}_2(L,t) &= \frac{E_1}{\alpha}\frac{d\hat{u}_2}{dx}(L) \\ &= \hat{A}_2 \frac{E_1 s}{c\alpha} \end{aligned}$$

To invert the transforms back into the time domain we will employ the DAC algorithm [6, 7, 8, 11] with Lanczos' sigma-factors [10, 13] smoothing out the Gibbs phenomenon present due to discontinuities in the wave front. The stress history at $x = L$ is shown in Figure 2.

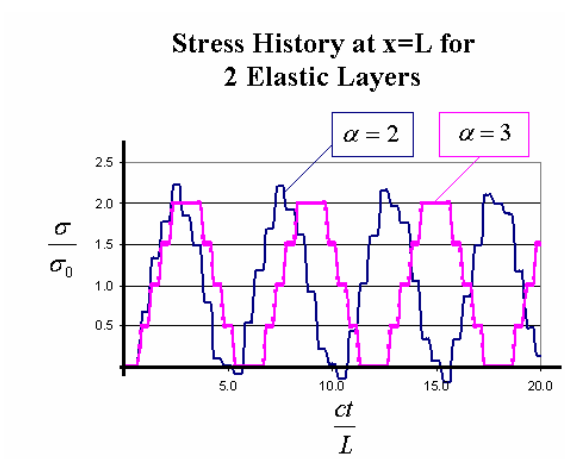

Figure 2: Stress histories measured at $x = L$ for two different composites. The peak stress is greater when $\alpha = 2$.

A brief examination of Figure 2 reveals that changing α had an effect on the peak stress experienced at $x = L$. When $\alpha = 2$ the stress amplitude reaches a peak of $2.23\sigma_0$, while the stress when $\alpha = 3$ is always between 0 and $2\sigma_0$. It is interesting to note that a homogeneous material, subject to the same boundary conditions, will always reach a peak stress amplitude of $2\sigma_0$ [9].

It's natural to ask if there are particular values of α for which the stress at the back of the composite is particularly low, especially lower than $2\sigma_0$ since this is the peak stress in a homogeneous strip. Unfortunately, the answer is no [15]. However, there are an infinite number of composites for which the peak amplitude is exactly $2\sigma_0$. Figure 3 presents the results of a brute force calculation of the peak stress measured at $x = L$ for different values of α.

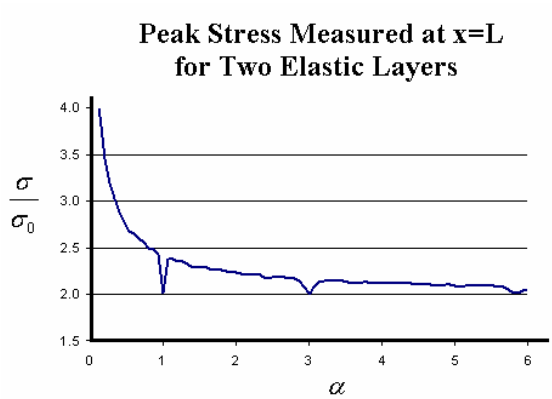

Figure 3: The peak stress for the two layer elastic composite as the design parameter α is varied.

In [15] it was shown that there are an infinite sequence of 'optimal' values of α for which the peak stress is exactly $2\sigma_0$. The sequence values are given by

$$\alpha_{opt} = \frac{1+\cos\left(\frac{\pi}{k}\right)}{1-\cos\left(\frac{\pi}{k}\right)}, \quad k = 2,3,\ldots$$

The first of these, $\alpha_{opt} = 1$, occurring when $k = 2$, corresponds to the homogeneous strip. The next two, $\alpha_{opt} = 3$ and $\alpha_{opt} \approx 5.8$, can be seen on Figure 3. That a sequence of optimal designs exists is not guaranteed. In fact, when considering strips subject to several time varying loads, impact from another rigid or elastic solid, and free or fixed back end conditions, only stress step inputs with a fixed back yielded an optimizable response [13].

In order to generalize the results for the case just considered, we extend our model in the next section. We develop a solution for a strip composed of an arbitrary number of

viscoelastic layers, subject to a stress step on the left and fixed on the right. Then in the following section we examine some simple methods for finding the optimal designs.

3 Physical Model and Solution

We limit our attention to a linear viscoelastic composite subject to a step in stress on one end and fixed at the far end, Figure 4.

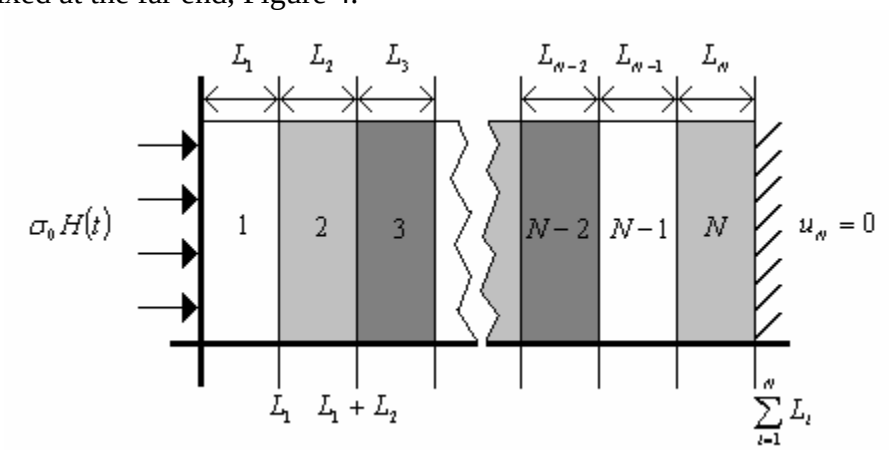

Figure 4: A viscoelastic composite subject to a suddenly applied stress on the left and fixed on the right.

Using a subscript to denote quantities that are layer dependent, the equations of motion are:

$$\rho_i \ddot{u}_i = \frac{\partial \sigma_i}{\partial x}, \quad i = 1,2,\ldots,N,$$

where ρ_i is density, u_i is displacement, σ_i is stress, and overdots denote differentiation with respect to time.

A linear viscoelastic solid has a constitutive relationship of the form

$$\sigma_i(x,t) = \int_0^t G_i(t-\tau)\frac{\partial \varepsilon_i}{\partial \tau}(x,\tau)d\tau ,$$

where ε_i is the linear strain and G_i is the relaxation modulus. Because each layer is homogeneous, the relaxation modulus is independent of position.

Assuming the composite is initially at rest we have the following ODE in the transformed space

$$s^2 \hat{u}_i = \frac{s\hat{G}_i}{\rho_i}\frac{d^2\hat{u}_i}{dx^2} .$$

We define

$$\hat{c}_i = \sqrt{\frac{s\hat{G}_i}{\rho_i}},$$

and the general solution in the transform space is

$$\hat{u}_i = \hat{A}_i \exp\left(\frac{sx}{\hat{c}_i}\right) + \hat{B}_i \exp\left(-\frac{sx}{\hat{c}_i}\right),$$

where $\hat{A}_i$ and $\hat{B}_i$ are constants (they depend on the transform variable s) whose values are determined from the boundary and interface conditions. The transformed stress is

$$\hat{\sigma}_i = \hat{A}\frac{s^2\hat{G}_i}{\hat{c}_i}\exp\left(\frac{sx}{\hat{c}_i}\right) - \hat{B}_i\frac{s^2\hat{G}_i}{\hat{c}_i}\exp\left(-\frac{sx}{\hat{c}_i}\right).$$

We will use a different, but equivalent, expression for the general solution in layer 1 (the layer subject to the stress), and layer N (the layer fixed on the right). The alternative expressions are chosen for convenience when incorporating the boundary conditions.

In layer 1 the displacement and stress are more conveniently expressed as

$$\begin{aligned}\hat{u}_1 &= \hat{A}_1 \cosh\left(\frac{sx}{\hat{c}_1}\right) + \hat{B}_1 \exp\left(-\frac{sx}{\hat{c}_1}\right)\\ \hat{\sigma}_1 &= \hat{A}_1\frac{s^2\hat{G}_1}{\hat{c}_1}\cosh\left(\frac{sx}{\hat{c}_1}\right) - \hat{B}_1\frac{s^2\hat{G}_1}{\hat{c}_1}\exp\left(-\frac{sx}{\hat{c}_1}\right)\end{aligned}.$$

Using the applied stress boundary condition at $x = 0$, we find that

$$\hat{B}_1 = -\frac{\hat{c}_1\sigma_0}{s^3\hat{G}_1}.$$

In layer N, the displacement and stress are most conveniently expressed as

$$\begin{aligned}\hat{u}_N &= \hat{A}_N \sinh\left(\frac{s}{\hat{c}_N}\left(x - \sum_{i=1}^{N} L_i\right)\right) + \hat{B}_N \exp\left(-\frac{sx}{\hat{c}_N}\right)\\ \hat{\sigma}_N &= \hat{A}_N\frac{s^2\hat{G}_N}{\hat{c}_N}\cosh\left(\frac{s}{\hat{c}_N}\left(x - \sum_{i=1}^{N} L_i\right)\right) - \hat{B}_N\frac{s^2\hat{G}_N}{\hat{c}_N}\exp\left(-\frac{sx}{\hat{c}_N}\right)\end{aligned}.$$

After application of the fixed boundary at $x = L$, we find that

$$\hat{B}_N = 0 .$$

Each layer has two undetermined constants from solving the 2nd order ODE, $\hat{A}_i$ and $\hat{B}_i$, $i = 1, \ldots, N$. Using the boundary conditions at $x = 0$ and $x = L$ we have found two of these, $\hat{B}_1$ and $\hat{B}_N$. The remaining $2(N-1)$ constants are determined using the continuity of displacement and stress at each of the $N-1$ interfaces within composite. The equations from the interface at $x = L_1$ are

$$\cosh\left(\frac{sL_1}{\hat{c}_1}\right)\hat{A}_1 - \exp\left(\frac{sL_1}{\hat{c}_2}\right)\hat{A}_2 - \exp\left(-\frac{sL_1}{\hat{c}_2}\right)\hat{B}_2 = \frac{\hat{c}_1\sigma_0}{s^3\hat{G}_1}\exp\left(-\frac{sL_1}{\hat{c}_1}\right)$$

$$\frac{s^2\hat{G}_1}{\hat{c}_1}\sinh\left(\frac{sL_1}{\hat{c}_1}\right)\hat{A}_1 - \frac{s^2\hat{G}_2}{\hat{c}_2}\exp\left(\frac{sL_1}{\hat{c}_2}\right)\hat{A}_2 + \frac{s^2\hat{G}}{\hat{c}_2}\exp\left(-\frac{sL_1}{\hat{c}_2}\right)\hat{B}_2 = -\frac{\sigma_0}{s}\exp\left(-\frac{sL_1}{\hat{c}_1}\right)$$

The equations from the 2nd through the $(N-2)^{nd}$ interface are

$$\exp\left(\frac{s\sum_{i=1}^{2}L_i}{\hat{c}_2}\right)\hat{A}_2 + \exp\left(-\frac{s\sum_{i=1}^{2}L_i}{\hat{c}_2}\right)\hat{B}_2 - \exp\left(\frac{s\sum_{i=1}^{2}L_i}{\hat{c}_3}\right)\hat{A}_3 - \exp\left(-\frac{s\sum_{i=1}^{2}L_i}{\hat{c}_3}\right)\hat{B}_3 = 0$$

$$\frac{s^2\hat{G}_2}{\hat{c}_2}\exp\left(\frac{s\sum_{i=1}^{2}L_i}{\hat{c}_2}\right)\hat{A}_2 - \frac{s^2\hat{G}_2}{\hat{c}_2}\exp\left(-\frac{s\sum_{i=1}^{2}L_i}{\hat{c}_2}\right)\hat{B}_2 - \frac{s^2\hat{G}_3}{\hat{c}_3}\exp\left(\frac{s\sum_{i=1}^{2}L_i}{\hat{c}_3}\right)\hat{A}_3 + \frac{s^2\hat{G}_3}{\hat{c}_3}\exp\left(-\frac{s\sum_{i=1}^{2}L_i}{\hat{c}_3}\right)\hat{B}_3 = 0$$

$$\vdots$$

$$\exp\left(\frac{s\sum_{i=1}^{N-2}L_i}{\hat{c}_{N-2}}\right)\hat{A}_{N-2} + \exp\left(-\frac{s\sum_{i=1}^{N-2}L_i}{\hat{c}_{N-2}}\right)\hat{B}_{N-2} - \exp\left(\frac{s\sum_{i=1}^{N-2}L_i}{\hat{c}_{N-1}}\right)\hat{A}_{N-1} - \exp\left(-\frac{s\sum_{i=1}^{N-2}L_i}{\hat{c}_{N-1}}\right)\hat{B}_{N-1} = 0$$

$$\frac{s^2\hat{G}_{N-2}}{\hat{c}_{N-2}}\exp\left(\frac{s\sum_{i=1}^{N-2}L_i}{\hat{c}_{N-2}}\right)\hat{A}_{N-2} - \frac{s^2\hat{G}_{N-2}}{\hat{c}_{N-2}}\exp\left(-\frac{s\sum_{i=1}^{N-2}L_i}{\hat{c}_{N-2}}\right)\hat{B}_{N-2} - \frac{s^2\hat{G}_{N-1}}{\hat{c}_{N-1}}\exp\left(\frac{s\sum_{i=1}^{N-2}L_i}{\hat{c}_{N-1}}\right)\hat{A}_{N-1} + \frac{s^2\hat{G}_{N-1}}{\hat{c}_{N-1}}\exp\left(-\frac{s\sum_{i=1}^{N-2}L_i}{\hat{c}_{N-1}}\right)\hat{B}_{N-1} = 0$$

The equations from the $(N-1)^{st}$ interface are

$$\exp\left(\frac{s\sum_{i=1}^{N-1}L_i}{\hat{c}_{N-1}}\right)\hat{A}_{N-1}+\exp\left(-\frac{s\sum_{i=1}^{N-1}L_i}{\hat{c}_{N-1}}\right)\hat{B}_{N-1}+\sinh\left(\frac{sL_N}{\hat{c}_N}\right)\hat{A}_N=0$$

$$\frac{s^2\hat{G}_{N-1}}{\hat{c}_{N-1}}\exp\left(\frac{s\sum_{i=1}^{N-1}L_i}{\hat{c}_{N-1}}\right)\hat{A}_{N-1}-\frac{s^2\hat{G}_{N-1}}{\hat{c}_{N-1}}\exp\left(-\frac{s\sum_{i=1}^{N-1}L_i}{\hat{c}_{N-1}}\right)\hat{B}_{N-1}-\frac{s^2\hat{G}_N}{\hat{c}_N}\cosh\left(\frac{sL_N}{\hat{c}_N}\right)\hat{A}_N=0$$

This is a linear system for the constants $\hat{A}_1,\hat{A}_2,\hat{B}_2,\ldots,\hat{A}_{N-1},\hat{B}_{N-1},\hat{A}_N$. Because all of these constants depend on the transform variable the system must be solved before the displacements and stresses can be inverted.

An example of a solution to the above equations follows. Consider composites consisting of 4, 8 and 16 viscoelastic layers. The relaxation modulus for each layer is of the form

$$G(t)=G_\infty+(G_0-G_\infty)e^{-\beta t}$$

Each constant in the relaxation modulus is layer dependent. The initial elastic response is G_0, the long term response is G_∞, and β is the decay parameter. The relaxation modulus gives the response of the solid to an applied and maintained unit strain. Figure 5 is a graph of this idealized response.

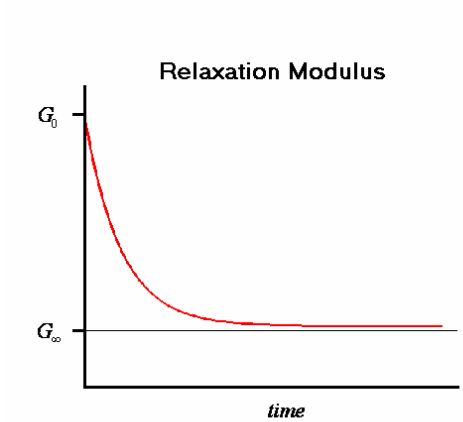

Figure 5: The Relaxation Modulus gives the stress within a viscoelastic solid experiencing a suddenly applied unit strain.

Using the solution outlined in this section we find the stress history, recorded at $x=L$. The results are graphed in Figure 6.

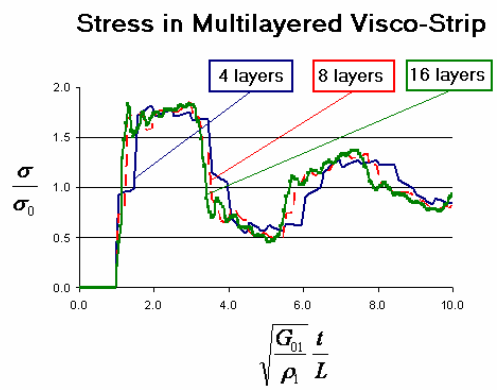

Figure 6: Stress history at $x = L$ in 3 multilayered composite strips. In each strip the initial elastic response alternates between E and $E/2$, the long time response is half the initial response, the decay parameter is 1 (in nondimensional units), and the density alternates between ρ and $\rho/2$.

For all of the composites in Figure 6, the initial elastic response alternates between E and $E/2$. The density also alternates between ρ and $\rho/2$. This ensures that the wave speed, c, is the same in each layer (analogous to the elastic strip in section 2). The long time response is half the initial response for each layer, and the decay parameter β is the same in each layer. The decay parameter has been chosen to be

$$\beta = \frac{1}{L}\sqrt{\frac{G_{01}}{\rho_1}} = \frac{1}{L}\sqrt{\frac{E}{\rho}}$$
$$= \frac{c}{L}$$

Because of the viscoelastic nature of the strips the peak stress is less that $2\sigma_0$ in each of the composites.

Another feature of each of the composites in Figure 6 is that each has equal amounts of two different materials. The difference between the composites is the number of layers into which those two materials are divided. As the number of layers increases, the time to the peak stress is lower. This would lead one to the conjecture that fewer layers are a better design in stress wave attenuators. This conjecture is consistent with the information we have regarding the homogeneous elastic solid, whose peak stress is always less than or equal an elastic composite's peak stress.

Figure 7 is another brute force calculation of the peak stress in a viscoelastic composite as a single design parameter is varied.

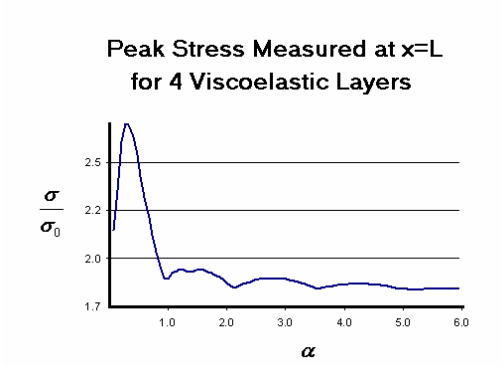

Figure 7: The peak stress for a 4 layer viscoelastic composite as the design parameter α is varied.

The material properties for the composites used to generate Figure 7 are presented in the following vectors.

$$L = \frac{L}{4}(1,1,1,1)$$

$$G_0 = \left(E, \frac{E}{\alpha}, E, \frac{E}{\alpha}\right)$$

$$G_\infty = \left(\frac{E}{2}, \frac{E}{2\alpha}, \frac{E}{2}, \frac{E}{2\alpha}\right)$$

$$\rho = \left(\rho, \frac{\rho}{\alpha}, \rho, \frac{\rho}{\alpha}\right)$$

$$\beta = \frac{1}{L}\sqrt{\frac{E}{\rho}}(1,1,1,1)$$

We can see local minimums occurring near $\alpha = 1$, 2, 3.5, and 5. The global minimum appears near $\alpha = 3.5$ (actual value is $\alpha = 3.5333$). In the next section we will consider methods that can find the optimal value of α without resorting to computationally burdensome, brute force calculations.

4 Methods of Optimization

Figure 3 is a curve that presents the maximum peak stress, experienced at the fixed end of a two layered elastic strip, for different values of the design parameter α. It was constructed by a brute force method where α was assigned a value, the numerical solution of the physical model was run and the peak stress recorded, then α was incremented and the process repeated. The increment was $\Delta\alpha = 1/15$, so the uncertainty in the computed optimal values of α is $\Delta\alpha$. However, we have the exact analytical results from [15] to verify our numerical results.

The brute force method has the advantage of reliability – if we choose the α increment small enough, we will always find the optimal designs. In section 2, when we

used the brute force method we needed to evaluate the peak stress for 90 different values of α, $0 < \alpha \le 6$, $\Delta\alpha = 1/15$. Typically, running the numerical solution long enough to capture the true peak stress takes a minute or two. For the sake of this argument, let's use the smaller value of one minute. Calculation of the peak stress for each design from $\alpha = 0$ to $\alpha = 6$, with an α increment of 1/15 takes 90 minutes.

What if we want to broaden the class of materials? If there are two design parameters the brute force procedure outlined above would require 8100 minutes, more than 5 ½ days. As design parameters are added and the computational burden of the physical model increased with viscoelastic layers, the need for a more efficient optimization algorithm is needed. In the next subsection we consider 3 optimization methods for a single design parameter. By necessity, all of these methods will be non-derivative based.

4.1 One Parameter Optimization

An examination of Figure 3 guides our choice of optimization methods. If we look at the minimum stress that occurs at $\alpha = 3$, we can see that the curve has a cusp-like appearance. This non-smooth behavior, especially at the minimum point, means that derivative based searches for the minimum will fail. Therefore, we seek algorithms that depend only on function evaluations. We describe 3 of these non-derivative searches in what follows. The advantage of these methods is that they are robust and reliable. The drawback is efficiency. Without the information that the derivative provides the algorithms must make more of the costly function evaluations.

The Fibonacci search [4] is the most basic of the one-parameter algorithms. Beginning with an interval $[a,b]$ that we know contains a single minimum, we divide the interval into thirds and make function evaluations at $a' = a + \frac{1}{3}(b-a)$ and $b' = a + \frac{2}{3}(b-a)$. If $f(a') < f(b')$ then we know that $[a,b']$ contains the minimum, otherwise $[a',b]$ contains the minimum. We divide the new interval into thirds and continue as before. When we stop the process we stop we choose the midpoint of the current interval as the approximation to the location of the minimum. This gives an error bound of half of the current interval length. That is, the error does not exceed

$$Err_{Fib} < \frac{1}{2}\left(\frac{b-a}{\lambda_n}\right),$$

Where n is the number of iterations of the algorithm and λ_n is the $(n+1)^{st}$ element of the Fibonacci sequence: $\lambda_0 = \lambda_1 = 1$ and $\lambda_n = \lambda_{n-1} + \lambda_{n-2}$ for $n \ge 2$.

Given an interval and an error tolerance we can calculate the number of iterations, N, of the Fibonacci search required to locate the minimum. Each iteration requires two function evaluations, so a total of $2N$ function evaluations will find the minimum.

For example, to find the minimum of the two layered elastic strip of section 2, with the same accuracy as the brute force algorithm, requires an error tolerance of 1/15.

If we have some a priori information that would lead us to believe that a minimum occurs between $\alpha = 2$ and $\alpha = 5$, then 7 iterations and 14 function evaluations are required to find the minimum. The computed minimum is $2.0297\sigma_0$, found at $\alpha = 3.0398$, well within 1/15 of the exact value of $\alpha = 3$.

A second algorithm is the Golden Search algorithm [4, 14]. The Golden Section Search algorithm is a modification of the Fibonacci search that reduces the function evaluations. Once the interval $[a,b]$ containing the minimum is determined, function evaluations are made at $a' = a + r^2(b-a)$ and $b' = a + r(b-a)$, where $r = \frac{1}{2}\left(\sqrt{5}-1\right)$ is the Golden Ratio. Again, if $f(a') < f(b')$ then $[a,b']$ contains the minimum. At the next iteration, function evaluations will be made at $a'' = a + r^2(b'-a)$ and $b'' = a + r(b'-a)$. However, $a' = b''$ so only one new function evaluation needs to be made. A similar situation occurs when $f(a') \geq f(b')$. Therefore, the first iteration requires two function evaluations, but each subsequent iteration only requires one more evaluation. Using the same error tolerance as in the brute force method we arrive at $\alpha = 2.9909$ with only 9 function evaluations.

A third algorithm is Brent's method, [2, 14]. This method uses a Golden Section Search until the interval has shrunk to a predetermined width. At this point the function is known at the endpoints of the small interval and one interior point of the interval. A parabola is fitted to these three points and the minimum of the parabola is located. The function is evaluated at this new point, giving a set of 4 points where the function is known. The 3 points with the lowest function values are kept, and the method continues. If the interval does not shrink (after multiple iterations), then it returns to the Golden Section Search until the interval is contracted and the parabolic interpolations continue.

Table 1 is a summary of the performance of these algorithms on the problem considered in section 2. In each case we began with the interval $[2,5]$ and the algorithm was terminated when the approximation was within 1/15 of the true minimum, using the algorithms' termination rule, not knowledge of the exact answer.

Table 1: Performance of several methods of optimization on the example problem of section 2, two elastic layers.

Method	Computed α_{opt}	Computed Peak Stress	Number of Function Evaluations
Exact	3.0	$2\sigma_0$	N/A
Brute Force	2.9333	$2.0066\sigma_0$	46
Fibonacci Search	3.0398	$2.0297\sigma_0$	14
Golden Section Search	2.9909	$2.0096\sigma_0$	9
Brent's Method	3.0096	$2.0087\sigma_0$	9

Clearly, the three optimization algorithms of this section outperform the brute force method. If there's anything surprising, it's that Brent's method appears to provide no advantage of the simpler Golden Section search.

Now, we apply each method to the 4 layer viscoelastic composite in Figure 7. We start each algorithm with the interval $[2,5]$, and use a tolerance of 1/15.

Table 2: Performance of the optimization algorithms on a 4 layer viscoelastic composite.

Method	α_{opt}	Computed Peak Stress	Number of Function Evaluations
Brute Force	3.5333	$1.7932\sigma_0$	46
Fibonacci Search	3.5158	$1.7837\sigma_0$	14
Golden Section Search	3.5319	$1.7919\sigma_0$	9
Brent's Method	3.5330	$1.7931\sigma_0$	10

Brent's method did a better job of matching the results of the brute force method. There's strong evidence that the true minimum is at $\alpha = 3.5333$, but there is no guarantee that this is the case.

Other results worth noting are first that despite more than one local minimum in the beginning interval, all the methods found the global minimum. Also, there is the question of the need for accuracy. We have reason to believe that Brent's method did the best at locating α_{opt}, but this came at the cost of an extra function evaluation. The relative difference between the Golden Search and Brent's method is 0.03%, hardly worth the extra effort of the function evaluation.

5 Conclusions

We've developed a method for finding the displacement, stress, or any other quantity that can be derived from these for a viscoelastic composite. The solid can have any number of layers, with any linear constitutive relationship between stress and strain. The composite is subject to a suddenly applied stress on one end and fixed on the other.

As the material properties of the layers are changed the composite experiences a different peak stress. In the situations we have considered there always appears to be designs, at least locally in the parameter space, that minimize the peak stress. For an elastic Goupillaud solid with only two layers there is an analytic formula for the optimal values of the design parameter. For all other situations we must rely on purely computational techniques to find the optimal designs.

Even with only a single design parameter a brute force approach to optimization is computationally prohibitive. Minimization algorithms that don't rely on derivatives are essential because of the cusp-like graphs for the peak stress. The Golden Section Search and Brent's method provide the most reliable results, but the simplicity and fewer function evaluations of the Golden Section search make it the most attractive of the algorithms tested.

Future studies need to address the multiparameter optimization problem. The single parameter optimization by brute force is a burden, but still a reasonable method of solution. When more parameters enter the problem the brute force method is no longer viable.

6 Acknowledgements

A majority of the research for this work was performed while Rich Laverty was a National Research Council Research Associate at the United States Military Academy and the US Army Research Laboratory.

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