Lower Confidence Bounds for the Probability of Non-Perforation in Homogeneous Armor

David W. Webb
U.S. Army Research Laboratory, Weapons and Materiels Research Directorate
Aberdeen Proving Ground, MD
webb@arl.army.mil

CDT Michael A. Faries
United States Military Academy, Company A3
West Point, NY
Michael.Faries@usma.edu

1 Executive Summary

The probability of non-perforation is an important performance measure for armor effectiveness. To estimate this probability in homogeneous armor, live-fire testing of an anticipated threat ammunition into semi-infinite armor may be conducted to record uncensored depths of penetration. Knowing the actual thickness of the armor, one can then obtain a point estimate for the probability of non-perforation using either a binomial (perforate/non-perforate) or a normal model of the penetration depths. Preferable to a point estimate, a lower confidence bound provides engineers with a worst-case estimate of the level of protection that the armor will provide. Applying the recent concept of generalized pivots (Weerahandi 1993) and assuming a normal distribution for the penetration depths, one can obtain a formula for the lower confidence bound for the probability of non-perforation. In this paper we develop this formula, and discuss the numerical method used to attain the solution in Mathematica®.

Keywords: lower confidence bound, generalized pivot, non-central t distribution, bisection method

2 Introduction

In evaluating the protective capability of armor systems, engineers and program managers are interested in estimating the probability of non-perforation probabilities based upon live-fire data. This paper is motivated by a hypothetical live-fire study in which projectiles are fired into semi-infinite targets of homogeneous armor to determine the depths of penetration. Semi-infinite armor is required since rounds which perforate the target have their penetration measurement censored at the thickness of the target, resulting in incomplete information on the penetrative capability of the round and the protective capability of the armor. Homogeneous armor is used so that the penetrations can assume a continuum of values that are modeled by a normal distribution.

For the following hypothetical example, suppose that fifteen projectiles are fired into semi-infinite homogeneous armor plates. The ensuing (unitless) depths of penetration are recorded in Table 1.

Table 1: Depths of Penetration into Homogeneous Armor (no units)

29.4	34.6	43.3	46.1	46.2
46.4	47.5	50.9	52.7	52.7
55.9	56.4	57.6	60.8	69.5

Assume that the production armor is proposed to be 60 units thick. To estimate the probability of non-perforation for future shots, one might simply count how many of these rounds had penetrations below 60 units and divide by the total number of shots. Using this very simplistic approach, one would arrive at non-perforation probability estimate of 13/15, or about 86.7%. However, this analysis leaves unanswered questions from both a statistical and an engineering viewpoint: What purpose does it serve to measure individual penetrations if the only matter of importance was whether or not the penetration exceeded 60 units? Is 86.7% the only reasonable estimate of non-perforation probability? Is it not possible for the actual probability to be 85%, or 80%, or even 75%? What is the smallest, or worst-case, estimate of non-perforation probability that would still be feasible?

A more complete solution to this estimation problem would be to provide a lower confidence bound on the probability of non-perforation at some acceptable level of confidence. It would also be preferable to use the actual penetration values in the determination of this interval, instead of reducing this continuous information (penetration depth) into a binomial response (perforation or non-perforation). Under the assumption of normally distributed penetration depths, we derive in this paper the desired lower confidence bound and discuss the numerical method used to attain the solution.

3 General Setup

Let X represent the penetration depth of a random projectile and x_0 denote the thickness of the production armor. We will assume that X is a normally distributed random variable with unknown mean μ and unknown standard variation σ. Using this notation, the probability of penetration is written as $P(X < x_0)$, and is known as the cumulative distribution function (CDF) evaluated at x_0. Notice that

$$P(X < x_0) = P\left(\frac{X-\mu}{\sigma} < \frac{x_0-\mu}{\sigma}\right) = P\left(Z < \frac{x_0-\mu}{\sigma}\right) = \Phi\left(\frac{x_0-\mu}{\sigma}\right), \tag{1}$$

where Z is a standard normal random variable, and $\Phi(\cdot)$ is the standard normal CDF commonly tabulated in statistics texts and available in many software packages.

Although the problem of point estimation of $P(X < x_0)$ from sample data has its origins in statistical quality control research in the 1950's, it was not until a paper by Owen and Hua (1977) that confidence interval estimators surfaced. Specifically, Owen and Hua derived lower confidence limits for $P(X \geq x_0)$, and described how to obtain both one-sided and two-sided limits for $P(X < x_0)$. Their paper is referenced by several other authors, and is the only one that offers a proof of their result. However, Owen and Hua's proof contains several errors in notation, and as a result is quite difficult to digest.

The following derivation of a lower confidence bound for $P(X < x_0)$ utilizes a totally different approach. We use the concepts of generalized pivots and generalized confidence intervals first described by Weerahandi (1993). In this framework, it is critical to distinguish between *X* and *x*. *X* refers to a random variable (usually a set of sufficient statistics) and *x* is the observed value of *X*. In general, the parameter for which a lower confidence bound is desired will be denoted by θ, while η will represent any other nuisance parameter(s) in the statistical model. To obtain a lower confidence bound for θ it is necessary to derive a generalized pivot for θ. A generalized pivot is a function of *X*, *x*, θ and η, and is written as $R(X; x, \theta, \eta)$. Since *R* is a function of the random quantity *X*, *R* is also a random quantity. By definition, the generalized pivot must satisfy the following two properties:

<u>Property 1</u>: $R(X; x, \theta, \eta)$ has a distribution that is free of any unknown parameters.
<u>Property 2</u>: The observed value of *R*, i.e., $r = R(x; x, \theta, \eta)$, is equal to θ.

The percentiles of a generalized pivot for θ are then used to obtain generalized confidence intervals for θ. Since their introduction in the 1980's, generalized pivotal quantities have been regarded as quite challenging to derive. Even Weerahandi (1993) states "… the construction of pivotals require some intuition". Iyer and Patterson's Substitution Method (unpublished), describes an algorithm helpful in constructing some generalized pivots.

4 Derivation of the Generalized Pivot

Step 1 of Iyer and Patterson's Substitution Method requires a set of sufficient statistics for the sample. In our case, this is the sample mean, $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, and the sample variance, $\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \overline{X}\right)^2$.

In Step 2, we require a same-sized set of statistics whose distributions are free of the unknown parameters. These are $Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}}$, which follows a normal distribution with

mean 0 and variance 1, and $V=(n-1)\frac{S^2}{\sigma^2}$, which is distributed as a chi-square random variable with $n-1$ degrees of freedom..

For Step 3, one solves for the unknown parameters in terms of the distribution-free statistics in Step 2. This results in $\sigma = S\sqrt{\frac{n-1}{V}}$ and $\mu = \overline{X} - ZS\sqrt{\frac{n-1}{nV}}$.

In Step 4, the unknown parameters appearing in θ are replaced by their equivalent expressions from Step 3. That is,

$$\theta = \Phi\left(\frac{x_0-\mu}{\sigma}\right) = \Phi\left(\frac{x_0 - \left(\overline{X} - ZS\sqrt{\frac{n-1}{nV}}\right)}{S\sqrt{\frac{n-1}{V}}}\right) = \Phi\left(\frac{x_0-\overline{X}}{S\sqrt{\frac{n-1}{V}}} + \frac{Z}{\sqrt{n}}\right).$$

Next in Step 5, the sufficient statistics are replaced by their observed values. This yields the random quantity $\Phi\left(\frac{x_0-\overline{x}}{s\sqrt{\frac{n-1}{V}}} + \frac{Z}{\sqrt{n}}\right)$.

In Step 6, the generalized pivot is finally obtained after the remaining random terms in Step 5 are replaced by their sufficient statistic-based equivalents. Therefore, the generalized pivot for $\theta = \Phi\left(\frac{x_0-\mu}{\sigma}\right)$ is given by

$$R = \Phi\left(\frac{x_0-\overline{x}}{s\sqrt{n-1}}\sqrt{(n-1)\frac{S^2}{\sigma^2}} + \frac{1}{\sqrt{n}}\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)$$

Property 1 can be shown to hold by noting that $R = \Phi\left(\frac{x_0-\overline{x}}{s\sqrt{n-1}}\sqrt{(n-1)\frac{S^2}{\sigma^2}} + \frac{1}{\sqrt{n}}\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)$ has the same distribution as the random quantity $\Phi\left(\frac{x_0-\overline{x}}{s\sqrt{n-1}}\sqrt{V} + \frac{1}{\sqrt{n}}Z\right)$, whose distribution is free of any unknown parameters.

To confirm Property 2, we note that the observed value of R is

$$r = \Phi\left(\frac{x_0 - \bar{x}}{s\sqrt{n-1}}\sqrt{(n-1)\frac{s^2}{\sigma^2}} + \frac{1}{\sqrt{n}}\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}\right) = \Phi\left(\frac{x_0-\bar{x}}{\sigma} + \frac{\bar{x}-\mu}{\sigma}\right) = \Phi\left(\frac{x_0-\mu}{\sigma}\right) = \theta .$$

With Properties 1 and 2 shown to hold, $R = \Phi\left(\frac{x_0 - \bar{x}}{s\sqrt{n-1}}\sqrt{(n-1)\frac{S^2}{\sigma^2}} + \frac{1}{\sqrt{n}}\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)$ is a generalized pivot for $\theta = \Phi\left(\frac{x_0-\mu}{\sigma}\right)$.

5 Lower Confidence Bound

The percentiles of a generalized pivot are used to obtain the limits of a confidence interval, or confidence bounds. In most cases, the complexity of the generalized pivot's distribution means that these percentiles must be obtained via Monte-Carlo simulation. However, in estimating the probability of non-perforation, the lower confidence bound can be expressed in closed form and Monte-Carlo simulation is not necessary.

By definition a $(1-\alpha)100\%$ lower confidence bound for $\theta = \frac{x_0-\mu}{\sigma}$ is that value *LCB* for which

$$1-\alpha = P(LCB \le R) \tag{2}$$

is satisfied.[1] Substituting the expression for *R*, we get

$$1-\alpha = P\left(LCB \le \Phi\left(\frac{x_0 - \bar{x}}{s\sqrt{n-1}}\sqrt{(n-1)\frac{S^2}{\sigma^2}} + \frac{1}{\sqrt{n}}\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)\right). \tag{3}$$

From Step 2 of the Substitution Method, $(n-1)\frac{S^2}{\sigma^2}$ and $\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ can be replaced by the previously defined distributions *V* and *Z*, which are free of unknown parameters. This yields

[1] It is implicit that the upper bound equals one in the probability statement of Equation 2, i.e., $1-\alpha = P(LCB \le R \le 1)$.

$$1-\alpha = P\left(LCB \le \Phi\left(\frac{x_0 - \bar{x}}{s\sqrt{n-1}}\sqrt{V} + \frac{1}{\sqrt{n}} Z \right)\right). \tag{4}$$

A series of algebraic rearrangements results in

$$1-\alpha = P\left(\frac{\bar{x} - x_0}{s/\sqrt{n}} \le \frac{Z - \sqrt{n}\Phi^{-1}(LCB)}{\sqrt{\frac{V}{n-1}}} \right) \tag{5}$$

The expression on the right side of the inequality is the definition of a non-central t distribution with $n-1$ degrees of freedom and non-centrality parameter $-\sqrt{n}\Phi^{-1}(LCB)$. Therefore,

$$1-\alpha = P\left(\frac{\bar{x} - x_0}{s/\sqrt{n}} \le T_{n-1,-\sqrt{n}\Phi^{-1}(LCB)} \right) \tag{6}$$

A non-central t random variable with non-centrality parameter $-\sqrt{n}\Phi^{-1}(LCB)$ is the mirror image of a non-central t random variable with non-centrality parameter $\sqrt{n}\Phi^{-1}(LCB)$ (Johnson and Kotz, 1970). Therefore,

$$1-\alpha = P\left(T_{n-1,\sqrt{n}\Phi^{-1}(LCB)} \le \frac{x_0 - \bar{x}}{s/\sqrt{n}} \right). \tag{7}$$

The probability expression of Equation 7 is the CDF of a non-central t random variable with $n-1$ degrees of freedom and non-centrality parameter $\sqrt{n}\Phi^{-1}(LCB)$ evaluated at $\frac{x_0 - \bar{x}}{s/\sqrt{n}}$; i.e.,

$$1-\alpha = G_{n-1,\sqrt{n}\Phi^{-1}(LCB)}\left(\frac{x_0 - \bar{x}}{s/\sqrt{n}} \right) \tag{8}$$

A lower confidence bound for $\theta = \Phi\left(\frac{x_0 - \mu}{\sigma} \right)$ is that value of *LCB*, which satisfies Equation 12. The solution to this equation must be obtained using numerical methods.

6 Bisection Method

Reconsider the data in Table 1, and the task of finding a 95% lower confidence bound for the probability of non-perforation for 60-unit thick armor, i.e., $P(X \le 60)$. These data should be first for normality before proceeding to apply Equation 8. Using Lilliefors Test (Conover 1980). The value of the test statistic is 0.1483, which corresponds to a P-value of 0.49. Therefore, the assumption of normality is not rejected and Equation 8 can be used.

The summary statistics for this data set are $\bar{x} = 50.00$ and $s = 10.00$. Therefore Equation 8 states that a 95% lower confidence bound for $P(X \le 60)$ is the solution to

$$.95 = G_{14,\sqrt{15}\Phi^{-1}(LCB)}\left(\frac{60-50}{10/\sqrt{15}}\right) = G_{14,\lambda}\left(\sqrt{15}\right);$$

or equivalently, we seek a zero for the function $H(\lambda) = G_{14,\lambda}\left(\sqrt{15}\right) - .95$. It can be shown that G is a monotonic decreasing function in the non-centrality parameter λ (see Appendix A), therefore so is H. To find the zero, we use any of several numerical methods for finding roots. The easiest is the bisection method, in which a small initial value, λ_{LO}, is chosen so that $H(\lambda_{LO}) > 0$,and a large initial value, λ_{HI}, is chosen so that $H(\lambda_{HI}) < 0$.

The functions $G_{14,\lambda}(x) - .95$ for $\lambda_{LO} = -10$ and $\lambda_{HI} = 10$ are shown in Figure 1. $H(\lambda)$ is determined by the intersecton of the the function $G_{14,\lambda}(x) - .95$ and the dotted vertical line $x = \sqrt{15}$. Note that $H(-10) > 0$ and $H(10) < 0$; so by the monotonicity of H, the value which satisfies $H(\lambda) = 0$ must lie between -10 and 10. Therefore, we average or "bisect" these two values to obtain the next guess, $\lambda_{MID} = \frac{\lambda_{LO} + \lambda_{HI}}{2} = \frac{-10+10}{2} = 0$.

As seen in Figure 2, $H(\lambda_{MID}) > 0$. Therefore the value which satisfies $H(\lambda) = 0$ must lie between 0 and 10. Therefore, the "new" λ_{LO} is set to 0, and we then bisect the updated λ_{LO} and λ_{HI} to obtain the next guess, $\lambda_{MID} = \frac{\lambda_{LO} + \lambda_{HI}}{2} = \frac{0+10}{2} = 5$.

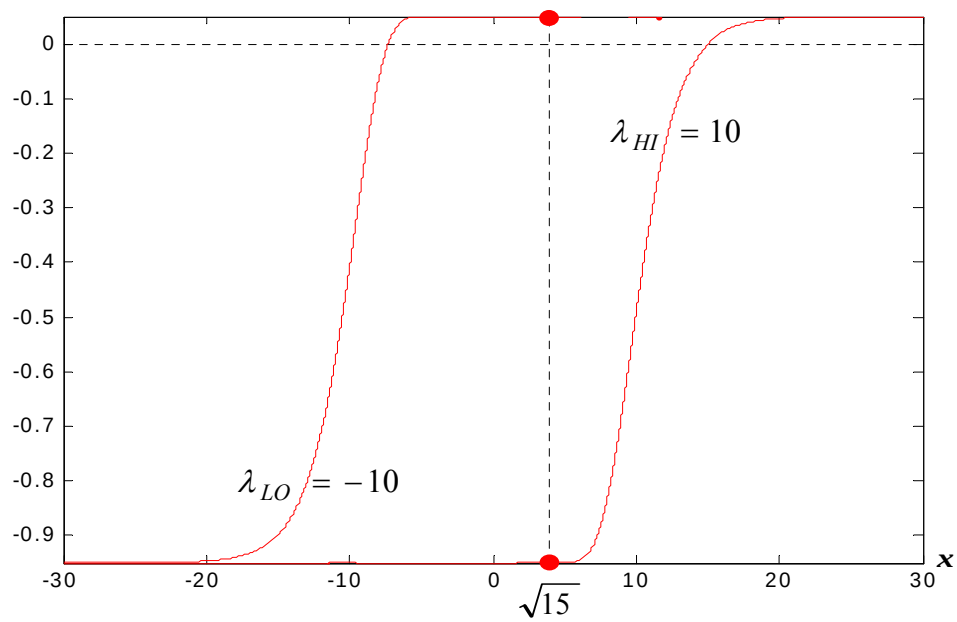

Figure 1: $G_{14,\lambda}(x)$ - .95 for initial values of non-centrality parameter

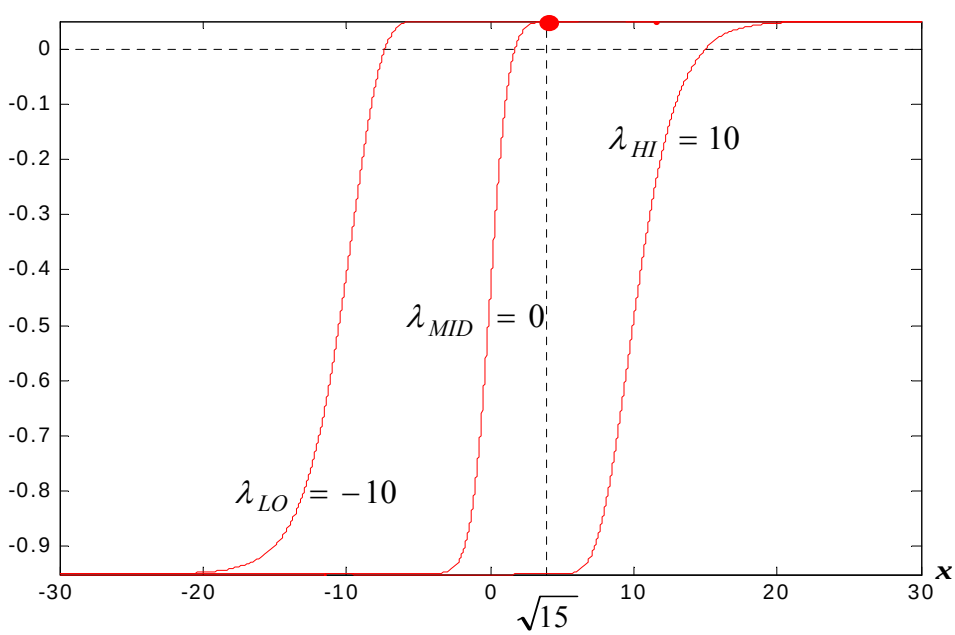

Figure 2: $G_{14,\lambda}(x)$ - .95 for first mid-value of non-centrality parameter

This method is repeated, closing in tighter upon the correct non-centrality parameter with each iteration. The algorithm ceases when $H(\lambda_{MID})$ is deemed sufficiently close to zero, more particularly when $|H(\lambda_{MID})| < \varepsilon$ for some predetermined extremely small positive ε. For the example data set, the algorithm converges to a non-centrality parameter of $\lambda = 1.7869$. The final step is to replace λ with $\sqrt{n}\Phi^{-1}(LCB)$, and to solve for *LCB*:

$$\sqrt{n}\Phi^{-1}(LCB) = 1.7869 \qquad \Leftrightarrow \qquad LCB = \Phi\left(\frac{1.7869}{\sqrt{n}}\right) = \Phi\left(\frac{1.7869}{\sqrt{15}}\right) = .6777$$

Finally, we report that the 95% lower confidence bound on the probability of non-perforation is about 67.77%.

7 Software

The bisection method for finding a lower confidence bound for the probability of non-perforation can be easily programmed into any mathematical software package capable of computing non-central t CDFs. A Mathematica ® program appears in Appendix B, for calculating a lower confidence bound for $\theta = \Phi\left(\frac{x_0 - \mu}{\sigma}\right)$. In this program the criteria used to stop the bisection method is not $|H(\lambda_{MID})| < \varepsilon$ but rather $\lambda_{HI} - \lambda_{LO} < \varepsilon$. This has negligible effect on the final determination of λ or LCB, but does reduce the number of calls to the non-central t CDF and speed execution of the code.

8 Summary

Using a generalized pivot, we have derived a lower confidence bound for the probability that a homogeneous armor plate of specified thickness will successfully stop a specific type of projectile from perforation. In the derivation of this interval estimate, we have assumed that the penetration depths are random and that they follow a normal distribution with unknown mean and variance. Reporting this lower confidence bound gives engineers and program managers a sense of the minimum level of protection afforded by the armor package against a particular threat.

The interval limit (or limits if a two-sided interval is desired) is a function of the sample mean and sample standard deviation of penetration depths, the number of shots fired, the thickness of the armor and percentiles of a non-central t distribution. However, since the non-centrality parameter associated with this distribution is a function of the interval limit, numerical methods are required to obtain the final solution. The Mathematica ® code used to perform this calculation is provided in the Appendix.

Confidence bands which graphically display the entire relationship between armor width and bounds on the probability of non-perforation follow naturally. These bands may be of help to armor designers in selecting an armor width that will provide a high degree of protection against specific enemy threats.

The beauty of the generalized approach is that it can be used to construct a generalized pivot for any function of the normal parameters, μ and σ. For example, a natural

extension of the work described in this paper is to derive interval estimates for the proportion of conformity, $P(a < X < b)$, using a generalized pivot. Recent research has shown that the repeated sampling properties of conventional confidence intervals/bounds may not necessarily hold for generalized confidence intervals/bounds. That is, some generalized pivots may yield generalized confidence intervals whose actual coverage probability is far below the nominal level of confidence. Monte-Carlo simulation of proposed generalized confidence interval/bounds is strongly recommended under a variety of parametric conditions to estimate the actual coverage probabilities.

9 List of Symbols Used

Scalar Values

n	number of observations in sample data
s	observed sample standard deviation
x_0	a specified thickness of armor
$\bar{x}$	observed sample mean
α	level of significance
σ	population standard deviation for a normally distributed random variable
θ	parameter for which an interval estimate is constructed; may actually be a function of one or more parameters
μ	population mean for a normally distributed random variable
η	a set of one or more nuisance parameters

Random Variables

$R(\ )$	generalized pivot; a function of random data, observed data, parameter of interest and perhaps other nuisance parameters
S	sample standard deviation of depths of penetration
T	student's t random variable
V	chi-square random variable
X	depth of penetration for a randomly selected projectile
$\bar{X}$	sample mean depth of penetration
Z	a standard normal random variable, with mean zero and variance one

Functions

$G(\ )$	student's t cumulative distribution function
$\Phi(\ )$	standard normal cumulative distribution function

Miscellaneous

LCB	lower confidence bound
$P(\ )$	probability of the parenthesized expression
Σ	summation

10 References

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11 Appendix A

Show that $H(\cdot)$ is a monotonic decreasing function. That is, show that if $\lambda_1 < \lambda_2$, then $H(\lambda_1) > H(\lambda_2)$.

<u>Proof:</u>

Let n, x_0, $\bar{x}$, s, α, λ_1, and λ_2 be fixed constants with $\lambda_1 < \lambda_2$. Let X_1 and Y be independent random variables, X_1 being normally distributed with mean λ_1 and variance 1, and Y following a chi-squared distribution with $n-1$ degrees of freedom. The support of X_1 is the set of all real numbers; and the support of Y is the set of all positive reals.

Now define $W = \sqrt{\frac{nY}{n-1}}\frac{x_0 - \bar{x}}{s}$. The support of W is either the set of all positive reals (if $x_0 \geq \bar{x}$), or the set of all non-positive reals (if $x_0 < \bar{x}$). However, in general, we will denote the support of W by Θ.

$P(X_1 < W)$ can be expressed as a double integral of the joint density of X_1 and W, namely,

$$P(X_1 < W) = \int_{\Theta}\int_{-\infty}^{w} f_{X_1}(x) f_W(w)\,dx\,dw. \tag{A1}$$

The double integral of Equation A1, can be rewritten as the sum of two double integrals:

$$P(X_1 < W) = \int_{\Theta}\int_{-\infty}^{w-(\lambda_2-\lambda_1)} f_{X_1}(x) f_W(w)\,dx\,dw + \int_{\Theta}\int_{w-(\lambda_2-\lambda_1)}^{w} f_{X_1}(x) f_W(w)\,dx\,dw. \tag{A2}$$

Each of these double integrals is positive, so by dropping the latter of the two, we obtain the inequality,

$$P(X_1 < W) > \int_{\Theta}\int_{-\infty}^{w-(\lambda_2-\lambda_1)} f_{X_1}(x) f_W(w)\,dx\,dw. \tag{A3}$$

But the double integral in Equation A3, represents a probability, namely $P(X_1 < W - (\lambda_2 - \lambda_1))$ which in turn equals $P(X_1 + \lambda_2 - \lambda_1 < W)$. So,

$$P(X_1 < W) > P(X_1 + \lambda_2 - \lambda_1 < W). \tag{A4}$$

Now define $X_2 = X_1 + (\lambda_2 - \lambda_1)$; then X_2 is normally distributed with mean λ_2 and variance 1. Therefore,

$$P(X_1 < W) > P(X_2 < W), \tag{A5}$$

Consider the standard normal random variable, Z, having mean zero and variance 1. Making use of the equalities $X_1 = Z + \lambda_1$ and $X_2 = Z + \lambda_2$, and the definition of W, we expand the above the result to obtain,

$$P\left(Z + \lambda_1 < \sqrt{\frac{nY}{n-1}}\,\frac{x_0 - \bar{x}}{s}\right) > P\left(Z + \lambda_2 < \sqrt{\frac{nY}{n-1}}\,\frac{x_0 - \bar{x}}{s}\right). \tag{A6}$$

Rearranging the expressions inside each of the probability statements,

$$P\left(\frac{Z + \lambda_1}{\sqrt{Y/n-1}} < \frac{x_0 - \bar{x}}{s/\sqrt{n}}\right) > P\left(\frac{Z + \lambda_2}{\sqrt{Y/n-1}} < \frac{x_0 - \bar{x}}{s/\sqrt{n}}\right). \tag{A7}$$

But by the definition of a non-central t random variable,

$$P\left(T_{n-1,\lambda_1} < \frac{x_0 - \bar{x}}{s/\sqrt{n}}\right) > P\left(T_{n-1,\lambda_2} < \frac{x_0 - \bar{x}}{s/\sqrt{n}}\right). \tag{A8}$$

Each of these probabilities is the cumulative non-central t distribution function evaluated at $\frac{x_0 - \bar{x}}{s/\sqrt{n}}$:

$$G_{n-1,\lambda_1}\left(\frac{x_0 - \bar{x}}{s/\sqrt{n}}\right) > G_{n-1,\lambda_2}\left(\frac{x_0 - \bar{x}}{s/\sqrt{n}}\right). \tag{A9}$$

Next we add the constant $\alpha - 1$, to both sides of the inequality:

$$G_{n-1,\lambda_1}\left(\frac{x_0 - \bar{x}}{s/\sqrt{n}}\right) + \alpha - 1 > G_{n-1,\lambda_2}\left(\frac{x_0 - \bar{x}}{s/\sqrt{n}}\right) + \alpha - 1. \tag{A10}$$

Using the definition of the function $H(\cdot)$, we obtain the desired result.

$$H(\lambda_1) > H(\lambda_2). \tag{A11}$$

12 Appendix B

Mathematica® Version 5.1 code for calculating a lower confidence bound for the probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$. User may supply his or her own data and values for alpha, x0, and epsilon.

```
(* Remove warning messages that do not affect results but clutter
output *)
Off[General::spell]; Off[General::spell1];
Off[NIntegrate::slwcon]; Off[NIntegrate::ncvb];

(* Enable use of statistical tools *)
<<Statistics`ContinuousDistributions`
<<Statistics`NormalDistribution`

(* Clear variables for use *)
Clear[data,n,noncent,x0,a,epsilon,lononcent,hinoncent,flag,K,ndist]

(* Define variables *)
alpha=.05; x0=115; epsilon =.000001;

(* Declare data *)
data={29.4, 34.6, 43.3, 46.1, 46.2, 46.4, 47.5, 50.9,
      52.7, 52.7, 55.9, 56.4, 57.6, 60.8, 69.5};

(* Calculate summary statistics and define K *)
n=Length[data];  xbar=Mean[data]//N;  sigma=StandardDeviation[data]//N;
K=(x0-xbar)/(sigma/Sqrt[n])//N;

(* Initialize flag and bounds on non-centrality parameter *)
flag=0; lononcent=xbar-10sigma//N; hinoncent=xbar+10sigma//N;

(* Execute bisection method *)
H:=CDF[NoncentralStudentTDistribution[n-1,noncent],K]+alpha-1
While[flag==0,noncent=Mean[{lononcent,hinoncent}];
      If[hinoncent-lononcent<epsilon,flag=1,
      If[H<0,hinoncent=noncent,lononcent=noncent]]]

(* Print lower confidence bound for probability of non-perforation *)
Print[TraditionalForm[StringForm["``0% LCB for P(non-perforation) =
``",
      100 (1-alpha),CDF[NormalDistribution[0,1],noncent/Sqrt[n]]]]]
```