Using Lie Symmetries to Study Differential Equations

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Introduction

We introduce the concepts of **Lie symmetries** and **center manifolds**.

Both Lie symmetries and center manifolds can be used to help solve differential equations.

We hope to use these mathematical ideas to aid scientists at the Natick Soldier Center who use differential equations in their research to support the Soldier in areas such as food (bacteria growth which affects shelf life), materials and protection equipment.

Lie Symmetries

Def. A Lie symmetry (LS) is a diffeomorphism that maps the set of solutions of a differential equation to itself.

In the plane, $\Gamma : (x, y) \mapsto (\hat{x}, \hat{y})$ is a LS if

$$\hat{y}' = \omega(\hat{x}, \hat{y}) \text{ when } y' = \omega(x, y)$$

where $\hat{y}' = \dfrac{D_x \hat{y}}{D_x \hat{x}}$ with D_x the total derivative

$$D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots.$$

An Example

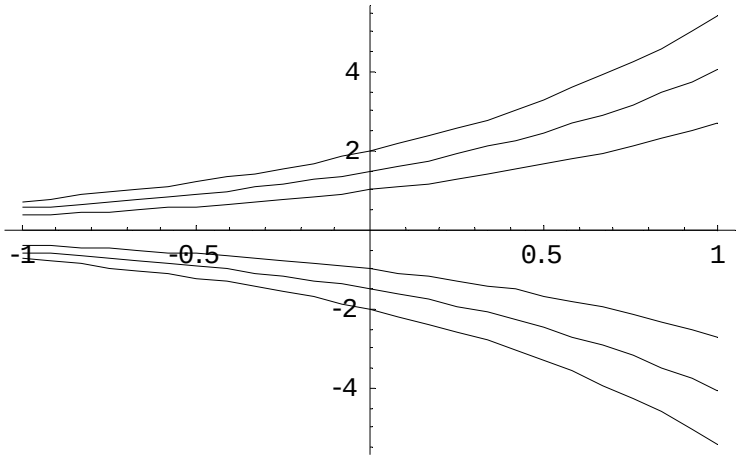
Consider the differential equation $y' = y$.

Define the LS: $\hat{x} = x + \varepsilon, \ \hat{y} = y$.

Solutions of $y' = y$ are $y = c_1 e^x$.

Under the symmetry:

$$\hat{y} = y = c_1 e^x = c_1 e^{\hat{x} - \varepsilon} = c_2 e^{\hat{x}}$$



Infinitesimals

Def. The infinitesimals of a 1-parameter symmetry $(\hat{x}, \hat{y})$ are

$$\xi = \xi(x, y) = \left.\frac{d\hat{x}}{d\varepsilon}\right|_{\varepsilon=0} \qquad \eta = \eta(x, y) = \left.\frac{d\hat{y}}{d\varepsilon}\right|_{\varepsilon=0}$$

The vector (ξ, η) is tangent to the orbit under the Lie symmetry of a point (x, y) on a solution curve.

Def. $X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ is called the infinitesimal generator of the symmetry.

Higher order ODEs

Extend the Lie symmetry $\Gamma : (x, y) \mapsto (\hat{x}, \hat{y})$

to the map

$$\Gamma : (x, y, y', \cdots y^{(n)}) \mapsto (\hat{x}, \hat{y}, \hat{y}', \cdots \hat{y}^{(n)})$$

$$\text{where } \hat{y}^{(k)} \equiv \frac{D_x \hat{y}^{(k-1)}}{D_x \hat{x}}.$$

Γ is a Lie symmetry of the differential equation

$$y^{(n)} = \omega(x, y, y', \cdots, y^{(n-1)}) \text{ if}$$

$$\hat{y}^{(n)} = \omega(\hat{x}, \hat{y}, \hat{y}', \cdots, \hat{y}^{(n-1)})$$

Determining Equations

We prolong the infinitesimal generator:

$$X^{(n)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \cdots + \eta^{(n)}\partial_{y^{(n)}}$$

$$\eta^{(1)}(x, y, y') = D_x\eta - y' D_x\xi$$

$$\eta^{(k)}(x, y, y', \ldots, y^{(k)}) = D_x\eta^{(k-1)} - y^{(k)} D_x\xi$$

For Γ to be a LS, the following must hold:

$$X^{(n)}(y^{(n)} - \omega(x, y, y', \ldots, y^{(n-1)})) = 0$$

This Linearized Symmetry Condition (LSC) gives a system of pdes in ξ and η. Solving this system gives our infinitesimal generators and thus our LS.

An Example

We find the symmetries of $y''=0.$

The LSC for this ODE is $\eta^{(2)}=0$ when $y''=0.$

$$\eta_{xx}+(2\eta_{xy}-\xi_{xx})y'+(\eta_{yy}-2\xi_{xy})y'^2-\xi_{yy}y'^3=0$$

Solving this results in

$$\xi(x,y)=c_1+c_3x+c_5y+c_7x^2+c_8xy,$$

$$\eta(x,y)=c_2+c_4y+c_6x+c_7xy+c_8y^2$$

We have 8 generators :

$$X_1=\partial_x, X_2=\partial_y,\ldots,X_7=x^2\partial_x+xy\partial_y, X_8=xy\partial_x+y^2\partial_y$$

Lie algebras

The infinitesimal generators of the LS form a Lie algebra.

The differential equation of our previous example gives rise to an 8-dimensional Lie algebra.

We can use the Lie algebra structure to help classify differential equations of particular orders.

A Lie algebra has a skew symmetric, bilinear bracket structure defined by:

$$[X_1, X_2] = X_1 X_2 - X_2 X_1$$

Classification of ODEs using Symmetry

For example, Lie classified all second order ODEs admitting 2-parameter symmetry groups into the following types:

1. $y'' = f(y')$
2. $y'' = f(x)$
3. $y'' = \frac{1}{x} f(y')$
4. $y'' = f(x) y'$

Invariant Solutions

Def. Solutions of a differential equation that are invariant under the action of the symmetry are invariant solutions.

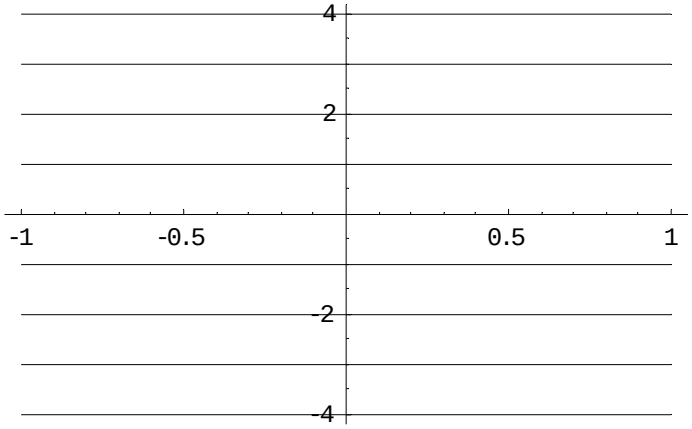
In this case, the orbit of a point on the solution curve is the solution curve.

Such a solution will satisfy $Q(x, y, y') = \eta - y'\xi$

Ex : $y' = 0$

with symmetry

$(\hat{x}, \hat{y}) = (x + \varepsilon, y)$



Differential Invariants

Def. Differential invariants are functions that are invariant under the (prolonged) action of the group generated by the infinitesimal generators.

A nonconstant function I is called a kth order differential invariant if

$$X^{(k)} I(x, y, y', \ldots, y^{(k)}) = 0.$$

We can reduce an ODE to an ODE consisting entirely of differential invariants.

$$\text{Example}: \; y^{(iv)} = \frac{2}{y}(1 - y')y''' \text{ is equivalent to } \frac{dv_3}{dr_3} = 1,$$

$$\text{with } r_3 = 2yy'' - y'^2, v_3 = y^2 y''', \text{ giving } y''' = \frac{2yy'' - y'^2 + c}{y^2}$$

Center Manifolds

Consider a system of differential equations

$$\dot{x} = f(x)$$

We can divide the eigenvalues of $Df(0)$ into 3 groups.

The eigenvalues with negative and positive real parts correspond to stable and unstable manifolds of the system.

From the eigenvalues with zero real part we can form the zero eigenspace. An invariant manifold tangent to this space is a center manifold (CM):

$$y = h(x) \text{ such that } h(0) = 0, h'(0) = 0$$

Center Manifolds cont'd

We write

$$\dot{x} = Ax + f(x, y)$$
$$\dot{y} = By + g(x, y)$$

If $f(0,0) = g(0,0) = 0,\ Df(0,0) = Dg(0,0) = 0$ and A has evals with zero real part, B with negative real part, we have a center manifold.

Then $h(x)$ is a center manifold if it satisfies the following:

$$h(0) = 0$$
$$h'(0) = 0$$
$$h'(x)(Ax + f(x, h(x))) = Bh(x) + g(x, h(x))$$

Center Manifolds cont'd

We find a center manifold to some suitable approximation by using

$$(M\varphi)(x) = \varphi'(x)(Ax + f(x, \varphi(x))) - B\varphi(x) - g(x, \varphi(x)).$$

Let $\varphi(x)$ be C^1 with $\varphi(0) = 0, \varphi'(0) = 0.$

Define $\varphi(x)$ such that $(M\varphi)(x) = O(x^q)$ some q.

Then defining

$$h(x) = \varphi(x) + O(x^q)$$

we get an approximation to a center manifold.

An Example

$$\dot{x} = xy + ax^3 + by^2 x$$

$$\dot{y} = -y + cx^2 + dx^2 y$$

Here $A = 0,\ \ B = -1,$

$$f(x, y) = xy + ax^3 + by^2 x,\ \ g(x, y) = cx^2 + dx^2 y,$$

Since $f(0,0) = g(0,0) = 0$ and $Df(0,0) = Dg(0,0) = 0$

there is a center manifold. We find that

$(M\varphi)(x) \to O(x^4)$, for $\varphi(x) = cx^2$.

Then $h(x) = cx^2 + O(x^4)$ is an approximation to a CM.

Links between Lie Symmetries and Center Manifolds

We ask what happens to manifolds under the action of the Lie symmetry.

Prop. [CG] All Lie point symmetries leave invariant both the stable and unstable manifolds of the dynamical system. Also, Lie point symmetries transform any center manifold into a center manifold (possibly the same).

Prop. [CG] Given any center manifold of a dynamical system, there is, generically, some nontrivial Lie point symmetry leaving invariant this center manifold, and conversely any Lie point symmetry of the dynamical system leaves invariant some center manifold.

LS and CM cont'd

Prop. [CG] The center manifold $y = h(x)$ remains invariant under the Lie symmetry $X = \varphi(x, y)\partial_x + \psi(x, y)\partial_y$ if and only if

$$\psi(x, h(x)) = (\partial_x h(x)) \cdot \varphi(x, h(x)).$$

Example
$$\dot{x}_1 = -x_1 r^2$$
$$\dot{x}_2 = -x_2 r^2$$
$$\dot{y} = -y$$
where $r^2 = x_1^2 + x_2^2$

The center manifold $h(x) = ce^{\frac{-1}{2r^2}}$ is invariant under $X = x_2\partial_{x_1} - x_1\partial_{x_2}$.

Future work

Scientists at the Natick Soldier Center use the following system to study growth and death kinetics in bacteria found in sandwiches:

$$\begin{pmatrix} \dot{M} \\ \dot{M}^* \\ \dot{A} \\ \dot{D} \end{pmatrix} = \begin{pmatrix} -k_1 M \\ k_1 M + M^*(G - \varepsilon A) \\ M^*(k_2 - \varepsilon A) \\ M^*(k_4 + \varepsilon A) \end{pmatrix} = \begin{pmatrix} -k_1 & 0 & 0 & 0 \\ 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M \\ M^* \\ A \\ D \end{pmatrix} + \begin{pmatrix} 0 \\ k_1 M^* - \varepsilon M^* A \\ k_2 M^* - \varepsilon M^* A \\ k_4 M^* + \varepsilon M^* A \end{pmatrix}$$

There is a center manifold because when we write the system in standard form we see that there are both negative and zero real part eigenvalues and the nonlinear function and its derivative are zero at the origin.

A Few References

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