Lower Confidence Bounds for the Probability of Non-Perforation in Homogeneous Armor

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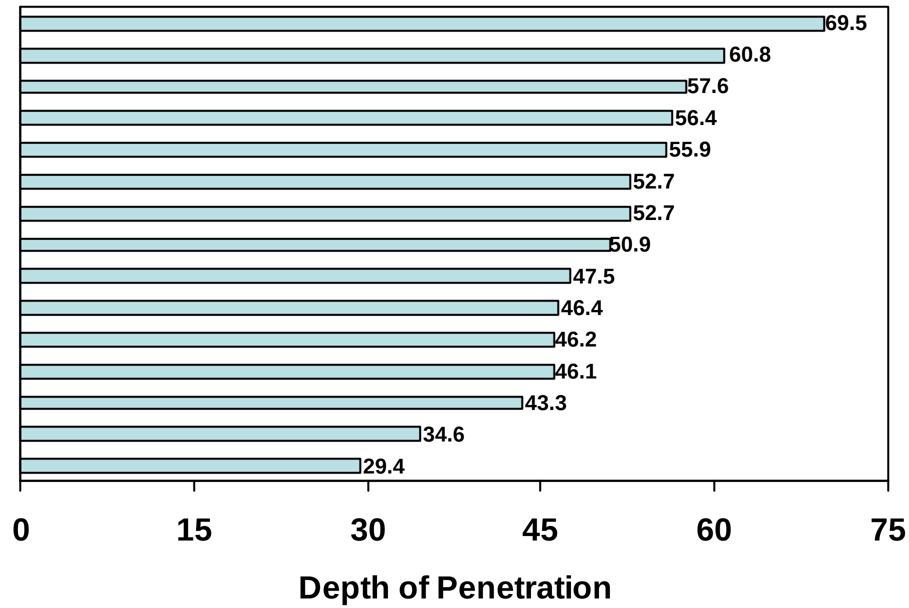

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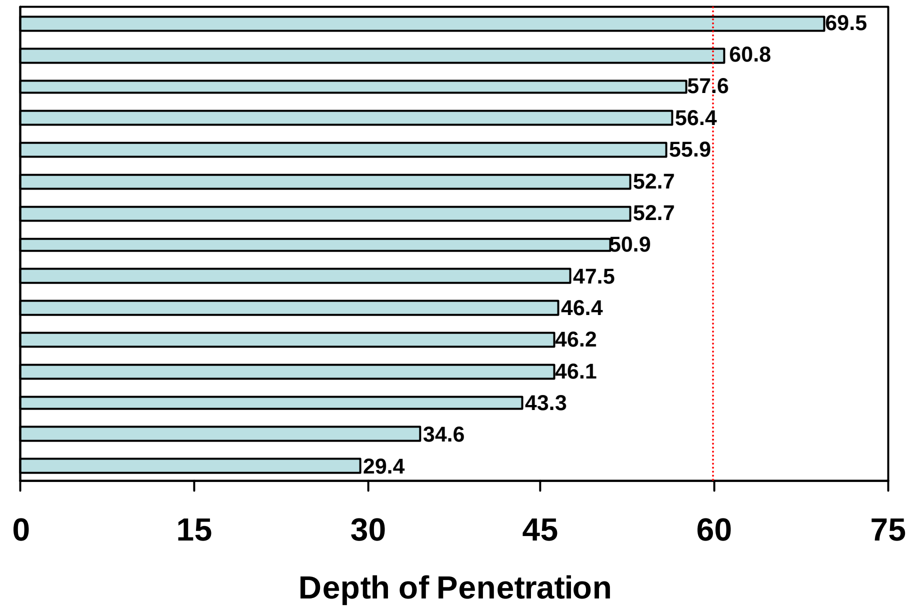

TASK: Find a 95% lower confidence bound (LCB) for the probability of non-perforation if the production armor is to be 60 units thick.

Let

X = penetration depth of a random projectile

x_0 = plate thickness (production)

where

$$X \sim N\left(\mu, \sigma^2\right)$$

$$P(non - perforation) = P(X < x_0)$$

$$P(non-perforation) = P(X < x_0)$$

$$= P\left(Z < \frac{x_0 - \mu}{\sigma}\right)$$

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$$= P\left(Z < \frac{x_0 - \mu}{\sigma}\right)$$

$$= \Phi\left(\frac{x_0 - \mu}{\sigma}\right)$$

Interval estimation of $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$ is not a new problem. See Patel & Read (1996), Hahn & Meeker (1990), Odeh and Owen (1980), or Owen and Hua (1977) for the solution.

But we will look at it from a completely different perspective: generalized pivots and generalized confidence intervals (Weerahandi, 1993).

Notation

X = Random variable, often a set of
sufficient statistics

x = Observed value of X

θ = Unknown parameter of interest

η = Nuisance parameter(s)

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A generalized pivot for θ must satisfy the following:

Property 1: $R(X; x, \theta, \eta)$ has a distribution that is free of any unknown parameters.

Property 2: The observed value, $r = R(x; x, \theta, \eta)$, equals θ.

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★ The percentiles of a generalized pivot for θ yield generalized confidence limits/bounds for θ. ★

In his seminal 1993 paper, Sam Weerahandi (the Godfather of Generalized Inference) even wrote :

"The problem of finding an appropriate pivotal quantity is a nontrivial task."

"... the construction of pivotals requires some intuition."

Finding a GCI for $\theta = \Phi\left(\dfrac{x_0 - \mu}{\sigma}\right)$ using Iyer and Patterson's Substitution Method

Step 1: Find a set of independent, sufficient statistics for the sample.

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \qquad\qquad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left(X_i - \bar{X}\right)^2$$

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Step 2: From these, find a same-sized set of statistics whose distributions are free of the unknown parameters.

$$Z = \frac{\overline{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1) \qquad\qquad V = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

Step 3: Solve for the unknown parameters in terms of the statistics in Step 2.

$$\mu = \overline{X} - ZS\sqrt{\frac{n-1}{nV}} \qquad \sigma = S\sqrt{\frac{n-1}{V}}$$

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$$\theta = \Phi\left(\frac{x_0 - \mu}{\sigma}\right) = \Phi\left(\frac{x_0 - \left(\overline{X} - ZS\sqrt{\frac{n-1}{nV}}\right)}{S\sqrt{\frac{n-1}{V}}}\right)$$

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Step 5: Substitute the (random) sufficient statistics with their observed values.

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Step 6: Substitute the remaining random terms with their sufficient-statistic based equivalent. Finally, this is the generalized pivot for θ !!!

$$R=\Phi\left(\frac{x_0-\bar{x}}{s\sqrt{n-1}}\sqrt{\frac{(n-1)S^2}{\sigma^2}}+\frac{1}{\sqrt{n}}\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)$$

In most cases, percentiles of the generalized pivot must be obtained via Monte-Carlo simulation. However, occasionally they may be expressed in closed form.

The latter can be shown to be the case for

$$R = \Phi\left(\frac{x_0 - \bar{x}}{s\sqrt{n-1}} \sqrt{\frac{(n-1)S^2}{\sigma^2}} + \frac{1}{\sqrt{n}} \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \right)$$

A (1-α)100% lower confidence bound for $\theta = \Phi\left(\frac{x_0 - \mu}{\sigma}\right)$ is that value "LCB" for which $1 - \alpha = P(LCB \le R)$

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$$= P\left(\Phi^{-1}(LCB) \le \frac{x_0 - \bar{x}}{s\sqrt{n-1}}\sqrt{V} + \frac{1}{\sqrt{n}}Z\right)$$

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$$= P\left(\frac{\bar{x}-x_0}{s/\sqrt{n}} \le \frac{Z-\sqrt{n}\Phi^{-1}(LCB)}{\sqrt{V/(n-1)}}\right)$$

(rearrange terms)

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$$= P\left(\frac{\bar{x} - x_0}{s/\sqrt{n}} \le T_{n-1,\,-\sqrt{n}\Phi^{-1}(LCB)} \right)$$

(definition of non-central t distribution)

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$$= P\left(T_{n-1,\,\sqrt{n}\Phi^{-1}(LCB)} \le \frac{x_0 - \bar{x}}{s/\sqrt{n}}\right)$$

($t_{n-1,\lambda}$ is mirror image of $t_{n-1,-\lambda}$)

$$1-\alpha = P\left(T_{n-1,\sqrt{n}\Phi^{-1}(LCB)} \le \frac{x_0-\bar{x}}{s/\sqrt{n}} \right)$$

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$$= G_{n-1,\sqrt{n}\Phi^{-1}(LCB)}\left(\frac{x_0 - \bar{x}}{s/\sqrt{n}} \right).$$

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The CDF of a non-central t distribution with n-1 degrees of freedom and non-centrality parameter $\lambda = \sqrt{n}\Phi^{-1}(LCB)$, evaluated at $\frac{x_0 - \bar{x}}{s/\sqrt{n}}$. This is the same result “proven” by Owen and Hua!

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The non-centrality parameter, λ, must be solved using numerical methods, e.g., the bisection method.

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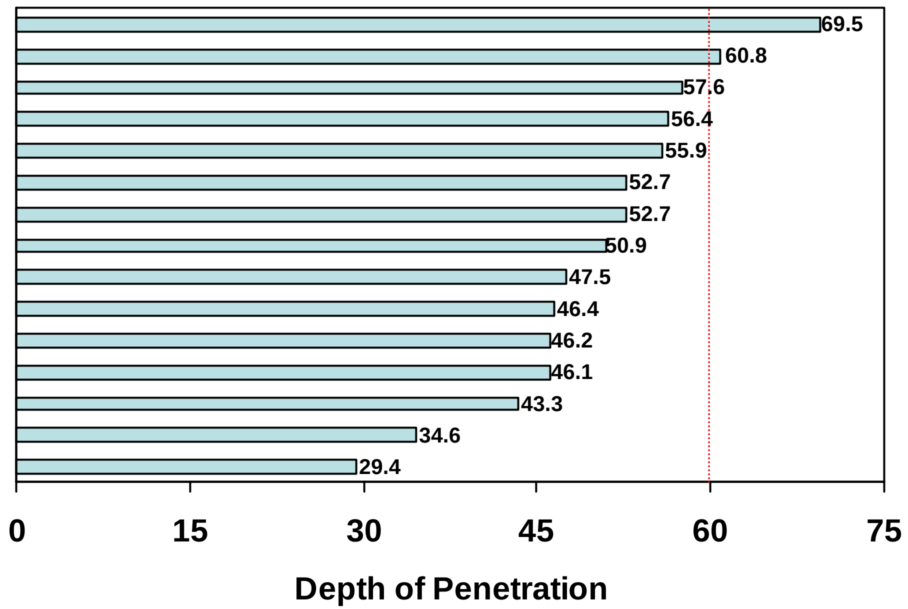

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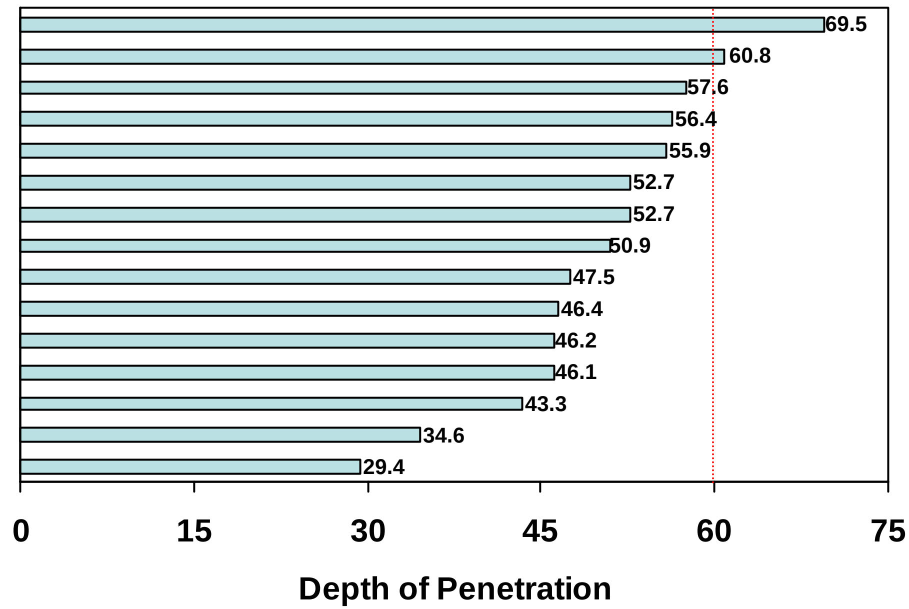

$\bar{x} = 50$ $\quad s = 10$

$n = 15$ $\quad \alpha = .05$ $\quad x_0 = 60$

$$1-\alpha = G_{n-1,\sqrt{n}\Phi^{-1}(LCB)}\left(\frac{x_0-\bar{x}}{s/\sqrt{n}}\right)$$

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$$.95 = G_{14,\,\sqrt{15}\,\Phi^{-1}(LCB)}\left(\frac{60-50}{10/\sqrt{15}}\right)$$

(substitute known values)

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$$.95 = G_{14,\,\lambda}\left(\sqrt{15}\right)$$

(reduce expressions)

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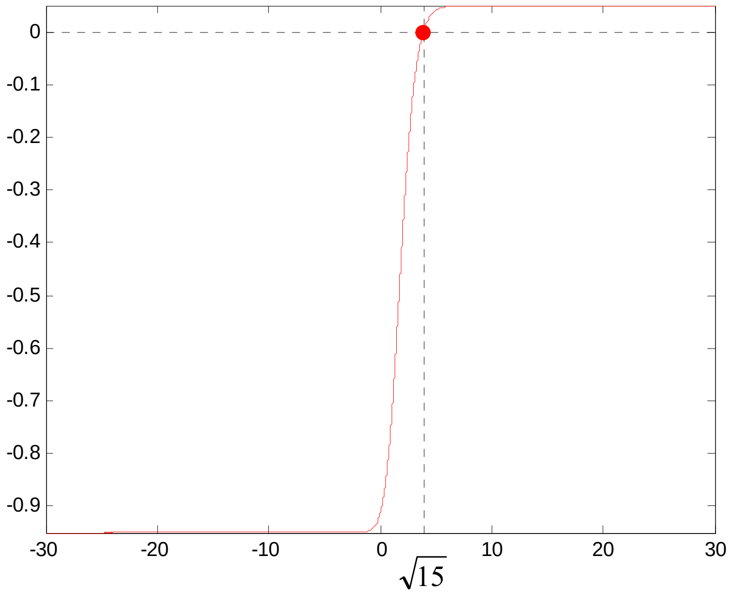
$$.95 = G_{14,\, \lambda}\left(\sqrt{15}\right)$$

$$0 = G_{14,\, \lambda}\left(\sqrt{15}\right) - .95$$

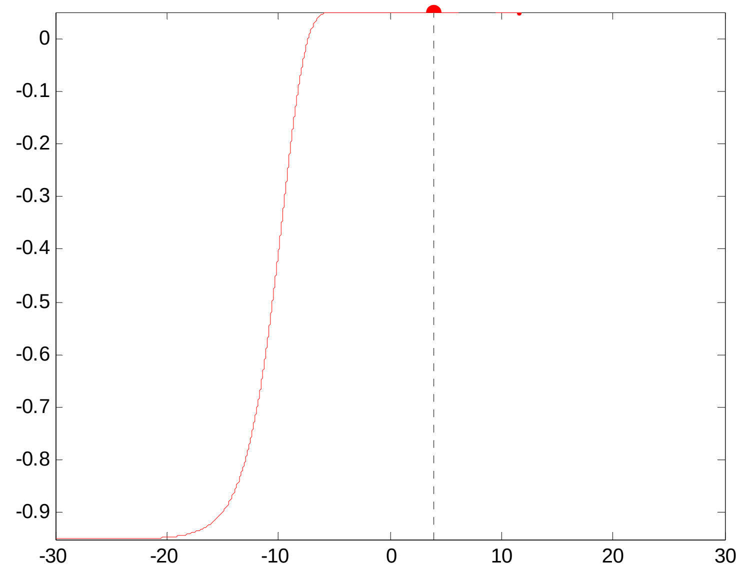
$$\text{We want } G_{14,\,\lambda}\left(\sqrt{15}\right) - .95 = 0\,!$$



$$\text{We will be satisfied with } \left|G_{14,\,\lambda}\left(\sqrt{15}\right) - .95\right| < \varepsilon\,.$$

When λ is "small", say λ_{LO}=-10

$$G_{14,\,\lambda_{LO}}\left(\sqrt{15}\right) - .95 > 0$$



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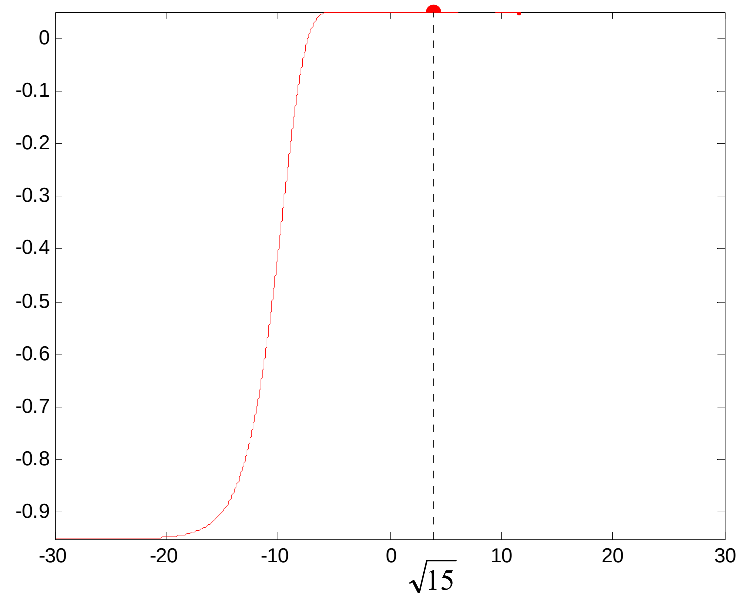
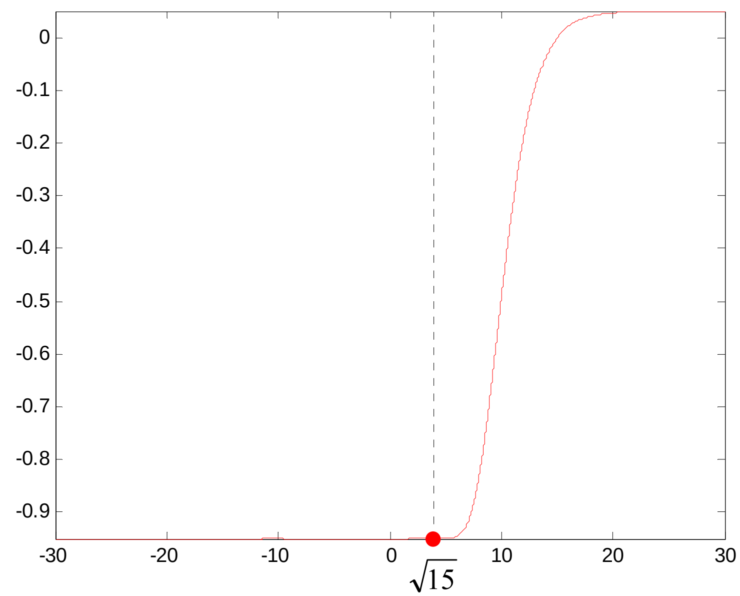
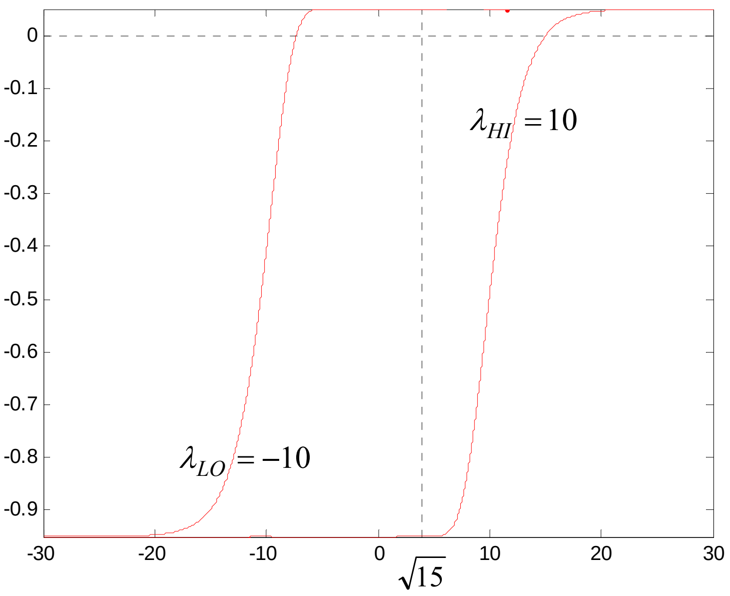


When λ is "large", say λ_{HI}=10

$$G_{14,\,\lambda_{HI}}\left(\sqrt{15}\right) - .95 < 0$$



$$\text{Take } \lambda_{MID} = \frac{\lambda_{LO} + \lambda_{HI}}{2} = 0$$



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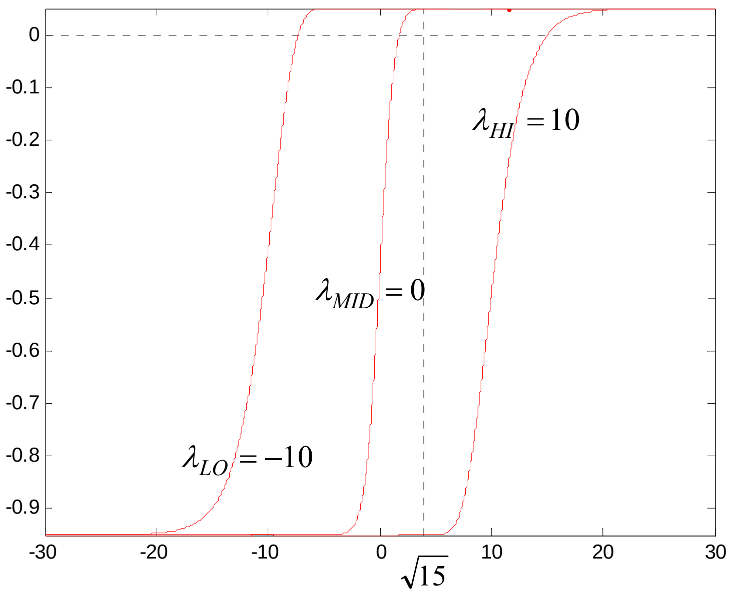

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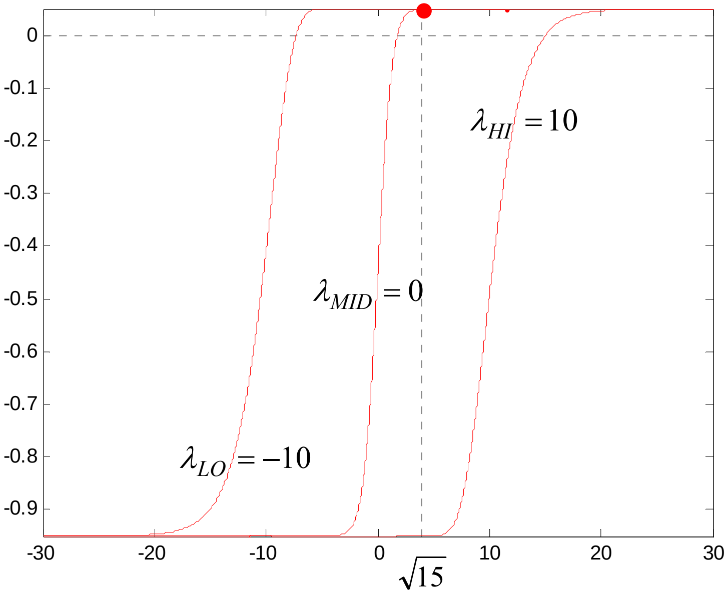
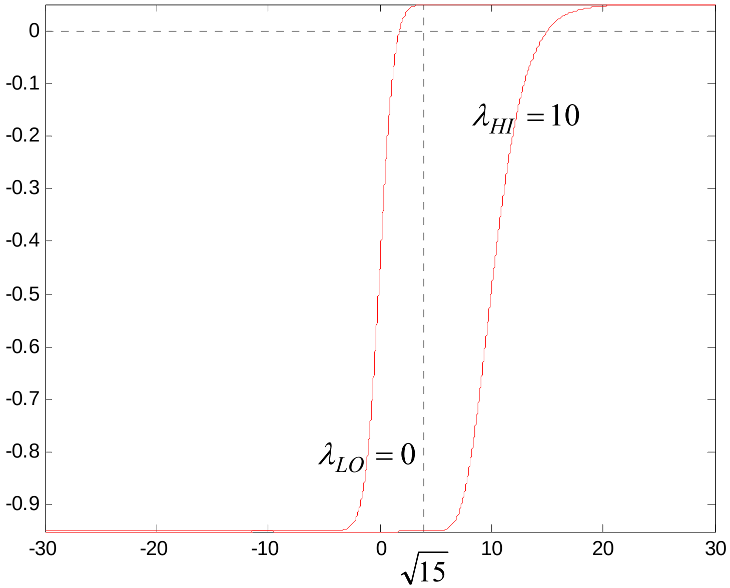


The “old” λ_{MID} now becomes the “new” λ_{LO} .



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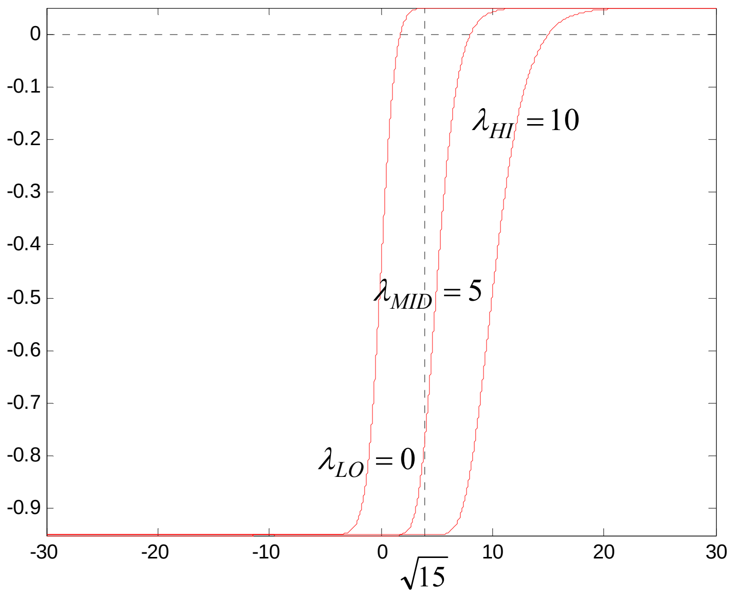

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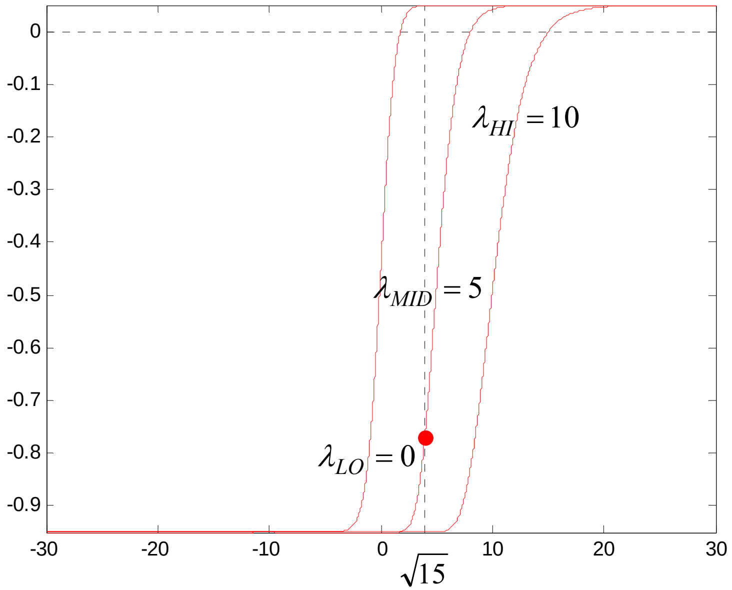
$$\text{Take } \lambda_{MID} = \frac{\lambda_{LO} + \lambda_{HI}}{2} = 0$$

$$G_{14,\,\lambda_{MID}}\left(\sqrt{15}\right) - .95 < 0$$



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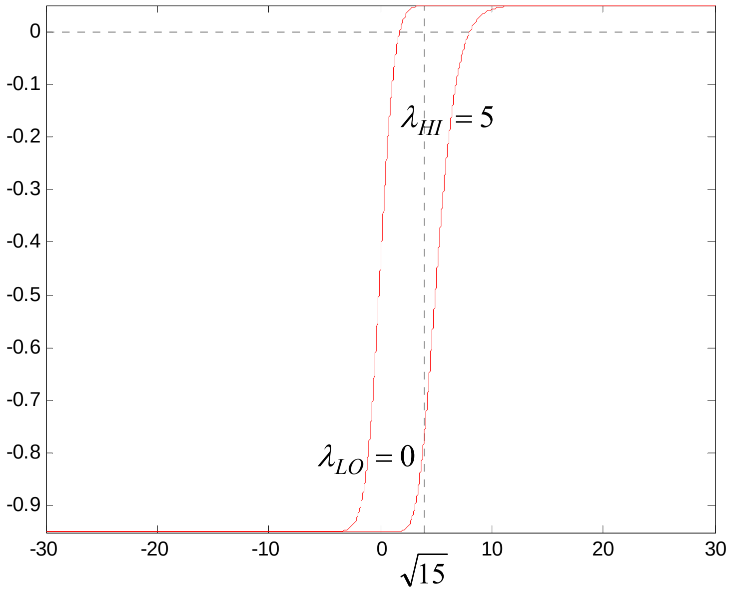
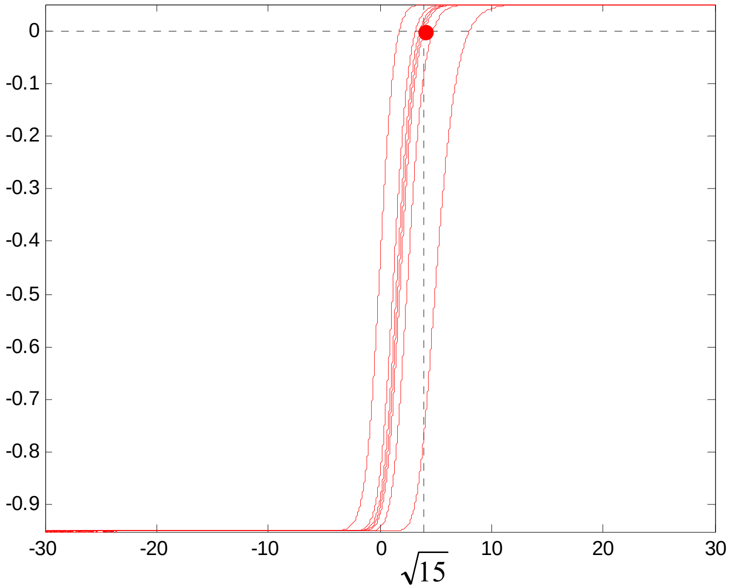


This "bisection method" continues until we converge upon a solution….



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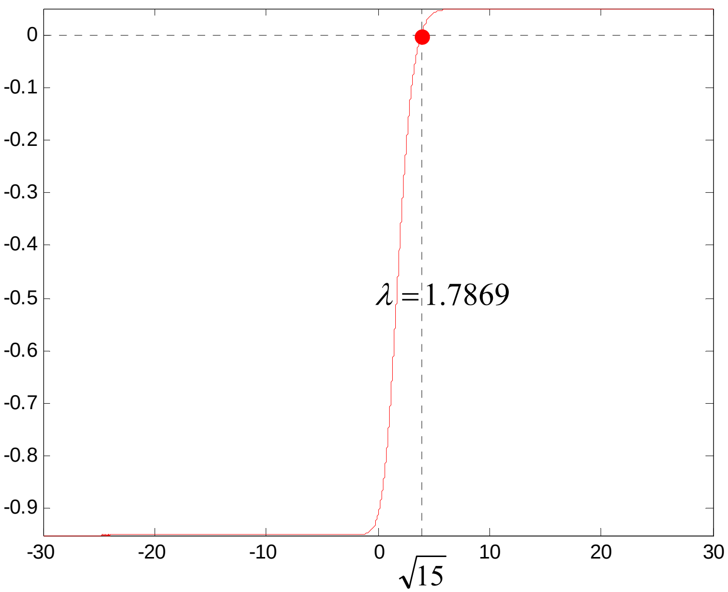

$$\lambda = 1.7869$$

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$$\sqrt{15}\,\Phi^{-1}\left(LCB\right) = 1.7869$$

(substitute non-centrality parameter)

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$$LCB = \Phi\left(1.7869 / \sqrt{15}\right)$$ (isolate unknown)

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$$LCB = \Phi\left(1.7869 / \sqrt{15}\right)$$

$$LCB = .6777$$ (FINAL ANSWER)

Some concluding remarks …

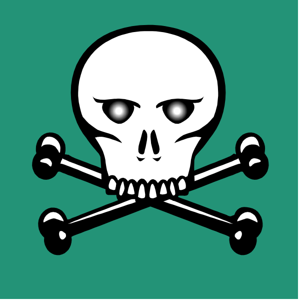
The beauty of this theory is that a generalized confidence interval can be constructed for ***any*** function of the normal parameters.

Ex. 1) Normal second moment (Weerahandi):

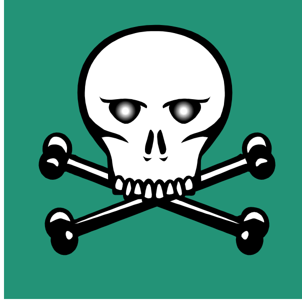
$$\theta = E(X^2) = \mu^2 + \sigma^2$$

Ex. 2) Interval estimation for the proportion of conformity (Iyer and Patterson):

$$\theta = P(a < X < b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$



CAVEAT EMPTOR!



-- The repeated sampling properties of conventional confidence intervals/bounds may not necessarily hold for generalized confidence intervals/bounds.

-- Monte-Carlo simulation of proposed generalized confidence intervals/bounds is strongly recommended under a variety of parametric conditions to estimate the actual coverage probabilities.

Important Reference Works

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Questions?