Using Symmetry and Dynamical System Methods to Study Differential Equations

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ABSTRACT

We calculate Lie symmetries and a center manifold for a set of nonlinear differential equations that model the growth-death kinetics of *Staphylococcus aureus* in foods. This is of interest to the food scientists at the US Army Natick Soldier Research Center. The goal of our study is to contribute to the results on the behavior of the system derived from numerical analysis. Lie symmetries can be a useful tool in the study of the behavior of a system that is not easily solved. The existence of a center manifold and the study of solutions restricted to it can provide insight into the overall behavior of the system. We link these two concepts by focusing in particular on the symmetries inherited by a center manifold.

1. Introduction

This article briefly introduces two important concepts in the study of differential equations and dynamical systems. We introduce the notion of a Lie symmetry, first on a single differential equation and then extend the idea to a system of first order differential equations. Symmetry methods were initially developed by Lie to aid in the classification of differential equations, but became more useful as they gave rise to methods of solving such equations. In a previous paper [DP] we focused on the different applications of Lie symmetries. Here we are more interested in finding the link between Lie symmetry and dynamical systems methods of studying differential equations. In our consideration of dynamical systems, we focus on the presence of invariant manifolds, and in particular, ask how Lie symmetries act on the invariant manifolds of a system. We briefly discuss results from the literature on the action of Lie symmetries on stable, unstable and center manifolds. After the introduction of Lie symmetries and dynamical systems in general, we include an application of these concepts to a system of differential equations arising in the work of food scientists at the Natick Soldier Research Center (SRC). These food scientists developed a system of equations to describe the life cycle of the bacteria *Staphylococcus aureus* as it occurs in medium moisture bread. Our goal in our consideration of these equations is to confirm results achieved through numerical

analysis by using symmetry and dynamical systems methods. We present our results and comment on the work that remains to be done on this project.

2. Lie Symmetries

Lie symmetries are diffeomorphisms that map the set of solutions of a differential equation to itself. These symmetries have several different uses in the study of differential equations. They can be used to write a system in terms of its canonical coordinates. Doing so may allow a researcher to write the system equivalently as a system of smaller order, possibly as a system that is directly solvable. Additionally, by writing systems in canonical coordinates, it may be possible to classify different systems as having the same structure or the same Lie symmetries, thus connecting seemingly unrelated systems of equations. These are interesting uses of Lie symmetries, ones we will consider in our continuing study of the Lie symmetries of differential equations. However, in this paper we discuss only how the Lie symmetries act on manifolds arising through dynamical systems theory. We introduce center manifolds below and state results on the invariance of these structures under Lie symmetries. Initially we present a brief introduction to Lie symmetries. The reader is referred to Hydon [H] for a more thorough treatment with examples.

2.1. **Lie symmetries of ODEs.** We begin our brief introduction to Lie symmetries by considering a diffeomorphism in the plane given by $\Gamma_\varepsilon : (x, y) \to (\hat{x}, \hat{y})$, with $(\hat{x}, \hat{y}) = (\hat{x}(x, y; \varepsilon), \hat{y}(x, y; \varepsilon))$. The map Γ_ε is actually an infinite family of maps with parameter ε. For the remainder of the paper we use the notation Γ to mean Γ_ε. We define the infinitesimals of this map, ξ and η, to be $\xi = \xi(x, y) = \left.\dfrac{d\hat{x}}{d\varepsilon}\right|_{\varepsilon=0}$ and $\eta = \eta(x, y) = \left.\dfrac{d\hat{y}}{d\varepsilon}\right|_{\varepsilon=0}$. If Γ is a Lie symmetry of a differential equation, $(\xi(x, y), \eta(x, y))$ is the tangent vector to the orbit of the points under the Lie symmetry at the point (x, y) on a solution curve of the differential equation. The operator $X = \xi\partial_x + \eta\partial_y$ is called the infinitesimal generator of the Lie symmetry given by $(\hat{x}, \hat{y})$. To find a diffeomorphism Γ that is Lie symmetry for a given differential equation, we calculate the infinitesimals and then use the definitions for ξ and η to find $\hat{x}$ and $\hat{y}$, integrating with respect to the parameter ε. What follows is the method of finding the infinitesimal generators.

We define the Lie symmetries of any ordinary differential equation (ODE)

$$y^{(n)} = \omega(x, y, y', \ldots, y^{(n-1)}), \tag{1}$$

with ω smooth, by extending a diffeomorphism $\Gamma : (x, y) \to (\hat{x}, \hat{y})$ to the map $\Gamma : (x, y, y', \ldots, y^{(n)}) \to (\hat{x}, \hat{y}, \hat{y}', \ldots, \hat{y}^{(n)})$. Define the $\hat{y}^{(k)}$ to be $\hat{y}^{(k)} = \frac{D_x \hat{y}^{(k-1)}}{D_x \hat{x}}$ using the total derivative $D_x = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots$. Any such map Γ will be a Lie symmetry of the differential equation (1) if it satisfies the equation, namely if $\hat{y}^{(n)} = \omega(\hat{x}, \hat{y}, \hat{y}', \ldots, \hat{y}^{(n-1)})$.

The functions ξ and η will be infinitesimals of the ODE (1) if they satisfy the linearized symmetry condition given by

$$\eta^{(n)} = \xi\omega_x + \eta\omega_y + \eta^{(1)}\omega_{y'} + \cdots + \eta^{(n-1)}\omega_{y^{(n-1)}} \tag{2}$$

where the $\eta^{(k)}$ are defined from the $\hat{y}^{(k)}$. These prolonged infinitesimals are calculated recursively by

$$\eta^{(k)}(x, y, y', \ldots, y^{(k)}) = D_x\eta^{(k-1)} - y^{(k)}D_x\xi.$$

We list the first few:

$$\begin{aligned}
\eta^{(1)} &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2; \\
\eta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 \\
&\quad + \{\eta_y - 2\xi_x - 3\xi_y y'\}y''; \\
\eta^{(3)} &= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 \\
&\quad - \xi_{yyy}y'^4 + 3\{\eta_{xy} - \xi_{xx} + (\eta_{yy} - 3\xi_{xy})y' - 2\xi_{yy}y'^2\}y'' \\
&\quad - 3\xi_y y''^2 + \{\eta_y - 3\xi_x - 4\xi_y y'\}y'''.
\end{aligned}$$

We can use the prolongation of the infinitesimal generator to rewrite the linearized symmetry condition. We find $X^{(n)} = \xi\partial_x + \eta\partial_y + \eta^{(1)}\partial_{y'} + \eta^{(2)}\partial_{y^{(2)}} + \cdots + \eta^{(n)}\partial_{y^{(n)}}$. Then (2) is equivalent to

$$X^{(n)}\left(y^{(n)} - \omega(x, y, y', \ldots, y^{(n-1)})\right) = 0. \tag{3}$$

Thus if we can find the infinitesimals ξ and η such that equation (3) holds, we can find a symmetry $\Gamma : (x, y, y', \ldots, y^{(n)}) \mapsto (\hat{x}, \hat{y}, \hat{y}', \ldots, \hat{y}^{(n)})$ of the differential equation (1). The condition (3) reduces to a system of PDE's in the partial derivatives of ξ and η which must be simultaneously solved to find the generators.

2.2. **Lie symmetries of a system of ODEs.** Since we can write any higher order ODE as a system of first order ODEs, it makes sense that we should be able to calculate Lie symmetries of a system of first order ODEs in an analogous way. We outline this below; this is the case in which we are interested because of the systems of equations arising in the work of scientists at the Natick SRC.

Given a system of first order ODEs

$$y_k' = \omega_k(t, y_1, y_2, \ldots, y_n), \quad k = 1, \ldots, n \tag{4}$$

we calculate the Lie symmetries that transform the variables $t, y_1, \ldots, y_n$ by finding the infinitesimal generators

$$X = \xi\partial_t + \eta_1\partial_{y_1} + \eta_2\partial_{y_2} + \cdots + \eta_n\partial_{y_n} \tag{5}$$

where $\xi = \xi(t, y_1, y_2, \ldots, y_n)$ and $\eta_k = \eta_k(t, y_1, y_2, \ldots, y_n)$ for all k. As in the case of a single ODE, the infinitesimals are derivatives of the images of the variables of the system with respect to the parameter of the Lie symmetry map. An infinitesimal generator must satisfy the linearized symmetry condition

$$X^{(1)}(y_k' - \omega_k(t, y_1, y_2, \ldots, y_n)) = 0, \quad k = 1, \ldots, n \tag{6}$$

when (4) holds. In this case the prolongation of X is as follows:

$$X^{(1)} = X + \eta_1^{(1)}\partial_{y_1'} + \eta_2^{(1)}\partial_{y_2'} + \cdots + \eta_n^{(1)}\partial_{y_n'} \tag{7}$$

where $\eta_k^{(1)}$ is defined to be $\eta_k^{(1)} = D_t\eta_k - y_k' D_t\xi$. The total derivative D_t is $D_t = \partial_t + y_1'\partial_{y_1} + \cdots + y_n'\partial_{y_n}$. Thus we have the following:

$$\eta_k^{(1)} = \partial_t\eta_k + y_1'\partial_{y_1}\eta_k + y_2'\partial_{y_2}\eta_k + \cdots + y_n'\partial_{y_n}\eta_k - y_k'(\partial_t\xi + y_1'\partial_{y_1}\xi + y_2'\partial_{y_2}\xi + \cdots + y_n'\partial_{y_n}\xi). \tag{8}$$

A system of first order ODEs has an infinite number of symmetries. We find the symmetries as in the case of higher order ODEs above, by finding the infinitesimals ξ, η_k that satisfy the linearized symmetry condition (6). Calculating symmetries by hand by finding their infinitesimal generators can be quite tedious since the symmetry condition

again reduces to a system of PDE's in the partial derivatives of ξ and η_k. Thus in our work below we found symmetries with the aid of the "Intro to Symmetry" package in Mathematica and a script included in [C].

3. Center Manifolds

By reducing a dynamical system to its lower dimensional center manifold, one can determine properties of the behavior of the whole system from that of the center manifold. In our study, the center manifold is the most important of the three possible invariant manifolds of a system. The stable, unstable and center manifolds are invariant, meaning that a solution curve starting on one of the three manifolds lies entirely on the same manifold.

To determine the three invariant manifolds, one takes a system of ODEs and writes it in normal form as follows:

$$\begin{aligned} \dot{x} &= Ax + g(x,y) \\ \dot{y} &= By + j(x,y) \end{aligned} \tag{9}$$

where A has eigenvalues with zero real parts, B has eigenvalues with nonzero real parts and the functions $g(x,y)$ and $j(x,y)$ are C^2. The stable and unstable manifolds are each unique, while a center manifold is not necessarily unique. The stable, unstable and center manifolds correspond to the eigenspaces of the eigenvalues with negative, positive and zero real parts respectively. All solutions of the system that decay toward the origin comprise the stable manifold; the unstable manifold consists of all solutions that grow away from the origin. A center manifold consists of solutions which do not exponentially decay toward or expand from the origin. If a system of differential equations can be written as above with at least one eigenvalue of the linear term with zero real part, then we are guaranteed the existence of a center manifold. See [Ca] or [GH] for more details, examples and a proof on the existence of center manifolds. The center manifold is an invariant manifold with $y = h(x)$, h smooth and $h(0) = Dh(0) = 0$. We approximate $h(x)$ by finding a function, to a particular order, that satisfies

$$(Mh)(x) = Dh(x)[Ax + g(x,h(x))] - Bh(x) - j(x,h(x)) = \overline{0}. \tag{10}$$

4. Lie Symmetries acting on Center Manifolds

Cicogna and Gaeta (see [CG99] for a more thorough treatment) have in recent years provided a link between the study of Lie symmetries and dynamical systems, and in particular, center manifolds. They have proven that the unstable and stable manifolds of a dynamical system will remain invariant under the action of any Lie symmetry. In this context we use the term invariant to mean that a manifold is mapped to itself by the Lie symmetry. Due to the non-uniqueness of center manifolds however, a center manifold may be mapped to another center manifold by a Lie symmetry. Another of their results states that for every center manifold there exists a Lie symmetry such that the center manifold is left invariant by that Lie symmetry. They have also shown that for every Lie symmetry there exists a center manifold that is left invariant by that symmetry. We make use of the following result that tells us precisely when a center manifold $h(x)$ is invariant under a given symmetry $X = \phi\partial_x + \psi\partial_y$.

Lemma 4.1 (Lemma 4 of [CG99] Chapter 7)**.** *A center manifold $h(x)$ is invariant under a Lie symmetry $X = \phi\partial_x + \psi\partial_y$ if and only if*

$$\psi(x, h(x)) = (\partial_x(h(x))) \cdot \phi(x, h(x)).$$

5. An Army application: the growth-death kinetics of *Staphylococcus aureus*

5.1. **The origin of the problem.** The system of equations we consider arose from the work of food scientists at the Natick SRC. These scientists were studying the growth of *Staphylococcus aureus* (*S. aureus*) that could occur in prepackaged sandwiches that the Army manufactures for its soldiers in the field. The goal of the scientists was to determine an accurate model for the growth of the bacteria. The food scientists use this model to predict the shelf-life of sandwiches so that food that is safe to eat is not prematurely discarded, while at the same any soldier who consumes a sandwich can be assured of its safety. The scientists found that all existing models for growth of bacteria in food considered only the growth or the death phase of the bacterial life cycle. They determined that this was not a sufficient way to model the bacteria; thus they constructed a model that included both the growth and death of the *S. aureus* bacteria cells. To find an accurate model, they collected data on bread crumbs with varying conditions of water activity, pH and temperature. The model they determined as the best fit to the data describes four phases in the life cycle of *S. aureus*. The model describes the bacteria cells passing through the stages of metabolizing (M), multiplying (M^*), and sensitization to death (M^{**}), and dead (D). Additionally, the scientists at the SRC hypothesized that there was an antagonist (A) present that would interact with the cells. With this assumption, the scientists were better able to find a model to fit their collected data. For a more thorough presentation of the model, the reader is referred to [RTDFK] and [TFRKD].

The first step in the process describes cells moving from lag phase to growth phase ($M \to M^*$). In the next step, cells multiply via binary division and then the newly multiplied cells interact with an antagonist ($M^* \to 2M^* + A$). The last two steps represent two different pathways to death: the first with cells interacting with an antagonist, then passing to sensitization before death ($A + M^* \to M^{**} \to D$) and lastly the cells experiencing natural death ($M^* \to D$).

These four phases are represented mathematically by the following system of ordinary differential equations with each phase having a rate constant (k_i) associated to it.

$$\dot{M} = -k_1 M \tag{11}$$

$$\dot{M}^* = k_1 M + M^*(G - \varepsilon A) \tag{12}$$

$$\dot{A} = M^*(k_2 - \varepsilon A) \tag{13}$$

$$\dot{D} = M^*(k_4 + \varepsilon A) \tag{14}$$

The constant $G = k_2 - k_4$ is the net natural growth rate and $\varepsilon = 10^{-9}k_3$ allows for a reparametrization. It is assumed that all the rate constants have non-negative values. The initial conditions at time zero are $M(0) = I$, the inoculum level $I \approx 10^3 - 10^4$, and $M^*(0) = A(0) = D(0) = 0$.

5.2. **A simplification.** As in the paper [DP], we see that the fourth equation is uncoupled since the variable D is not present in any of the first three equations and $\dot{D}$ can be determined from M^* and A. Therefore to investigate the dynamics of the system, we can reduce to a system of three equations. Renaming the variables ($y_1 = M,\ y_2 = M^*,\ y_3 = A$) we find the following system equivalent to equations (11)-(14):

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & G & 0 \\ 0 & k_2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -\varepsilon y_2 y_3 \\ -\varepsilon y_2 y_3 \end{pmatrix} \tag{15}$$

In order to calculate a center manifold, we must write the system in normal form; we do this now. First note that the eigenvalues of the linear terms of the system are $-k_1$, G, and zero. To normalize the system we need to consider 2 cases: $G \neq 0$ and $G = 0$. First we consider $G \neq 0$ and let the matrix T consist of eigenvectors of the associated eigenvalues. Then we perform a change of variable operation by solving the following for the new variables u, v, w:

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & G + k_1 \\ 0 & 1 & -k_1 \\ 1 & \frac{k_2}{G} & k_2 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{16}$$

After solving for u, v, w and taking derivatives we arrive at the normalized system:

$$u' = 0 \cdot u + \left(1 - \frac{k_2}{G}\right) f(u, v, w) \tag{17}$$

$$\begin{pmatrix} v' \\ w' \end{pmatrix} = \begin{pmatrix} G & 0 \\ 0 & -k_1 \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} + \begin{pmatrix} f(u, v, w) \\ 0 \end{pmatrix} \tag{18}$$

where $f(u, v, w) = -\frac{\varepsilon}{G}(v - k_1 w)(k_2 v + G(u + k_2 w))$. Since $f(u, v, w)$ and its partials with respect to $u,\ v$, and w are all zero at the origin $(u, v, w) = (0, 0, 0)$, the system is in normal form.

Now considering the case $G = 0$, rewrite the simplified system as

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = \begin{pmatrix} -k_1 & 0 & 0 \\ k_1 & 0 & 0 \\ 0 & k_2 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -\varepsilon y_2 y_3 \\ -\varepsilon y_2 y_3 \end{pmatrix} \tag{19}$$

Note that in this case there are two zero eigenvalues and one negative eigenvalue for the matrix of the linear term. Since zero is a repeated eigenvalue, we use the generalized eigenvectors to find the normalization of this system. The generalized eigenvectors are $(0, 0, 1)$, $(0, 1, 0)$, and $(1, -1, \frac{k_2}{k_1})$. To transform the system, again let T be the matrix consisting of these eigenvectors and let

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & \frac{k_2}{k_1} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}. \tag{20}$$

As above, this allows us to write the system in normal form:

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & k_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} g(u, v, w) \\ g(u, v, w) \end{pmatrix} \tag{21}$$

$$w' = -k_1 w + 0 \tag{22}$$

where $g(u,v,w) = -\varepsilon(v-w)(u+\frac{k_2}{k_1}w)$. Since $g(u,v,w)$ is zero at the origin and all of its first partial derivatives are also zero at the origin, again the system is in normal form.

For the remainder of this paper, we will consider the growth-death model using the variables u, v, w.

5.3. **Lie Symmetries of the growth-death model.** We calculate the Lie symmetries of the system described by (17) and (18) for the case $G \neq 0$ noting that the sign of G does not affect the results. Then we use (21) and (22) for the case $G = 0$. As noted previously, there are infinitely many symmetries of this system because it is a system of first order ODEs. Computing power limits the number and order of Lie symmetries we can calculate. In both cases, we list symmetries up to order three in the variables u, v, w. We write the infinitesimal generators $X = \xi\partial_t + \eta_1\partial_u + \eta_2\partial_v + \eta_3\partial_w$ as $X = \{\xi, \eta_1, \eta_2, \eta_3\}$:
Case $G \neq 0$:

$$
\begin{aligned}
X_1 &= \{1,0,0,0\} \\
X_2 &= \{j(u,v,w), \frac{1}{G}(G-k_2)j(u,v,w)f(u,v,w), j(u,v,w)(Gv+f(u,v,w)), -k_1wj(u,v,w)\} \\
X_3 &= \{l(u,v,w), \frac{1}{G}(G-k_2)l(u,v,w)f(u,v,w), l(u,v,w)(Gv+f(u,v,w)), -k_1wl(u,v,w)\} \\
X_4 &= \{0, \frac{1}{Gk_1}(G-k_2)f(u,v,w), \frac{1}{k_1}(Gv+f(u,v,w)), -w\} \\
X_5 &= \{\frac{t}{k_1}, \frac{t}{Gk_1}(G-k_2)f(u,v,w), \frac{t}{k_1}(Gv+f(u,v,w)), -tw\} \\
X_6 &= \{m(u,v,w), \frac{1}{G}(G-k_2)m(u,v,w)f(u,v,w), m(u,v,w)(Gv+f(u,v,w)), -k_1wm(u,v,w)
\end{aligned}
$$

where $f(u,v,w)$ is as above, $j(u,v,w) = v - k_1w$, $l(u,v,w) = u + \frac{k_2}{G}v + k_2w$ and $m(u,v,w) = -\frac{1}{\varepsilon}(G+k_1)w$.
Case $G = 0$:

$$
\begin{aligned}
X_1 &= \{1,0,0,0\} \\
X_2 &= \{v-w, \frac{1}{k_1}(v-w)n(u,v,w), -\varepsilon(v-w)p(u,v,w), -k_1(v-w)w\} \\
X_3 &= \{u+\frac{k_2}{k_1}w, \frac{1}{k_1}(u+\frac{k_2}{k_1}w)n(u,v,w), -\varepsilon(u+\frac{k_2}{k_1}w)p(u,v,w), -k_1(u+\frac{k_2}{k_1}w)w\} \\
X_4 &= \{0, \frac{-1}{\varepsilon k_1}n(u,v,w), p(u,v,w), \frac{k_1}{\varepsilon}w\} \\
X_5 &= \{\frac{-t}{\varepsilon}, \frac{-1}{\varepsilon k_1}tn(u,v,w), tp(u,v,w), \frac{k_1}{\varepsilon}tw\} \\
X_6 &= \{\frac{-1}{\varepsilon}w, \frac{-1}{\varepsilon k_1}wn(u,v,w), wp(u,v,w), \frac{k_1}{\varepsilon}w^2\}
\end{aligned}
$$

where $n(u,v,w) = \varepsilon k_2 w(-v+w) + k_1(k_2v + \varepsilon u(-v+w))$ and $p(u,v,w) = (v-w)(u+\frac{k_2}{k_1}w)$.

5.4. **Center manifolds computations on the growth-death model.** Considering the model given in equations (17) and (18), it can be seen immediately that the system has a one-dimensional center manifold in the case $G \neq 0$. For $G > 0$, there are also a one-dimensional stable manifold and a one-dimensional unstable manifold. In the

case $G < 0$, in addition to the center manifold, there is only a two-dimensional stable manifold.

For the case $G \neq 0$, the sign of G does not change the outcome of the center manifold calculation. Here we calculate a center manifold $h : V \to \mathbb{R}^2$, with $V \subset \mathbb{R}$ a neighborhood of the origin. Thus let $h(x) = (h_1(x), h_2(x)) = (ax^2 + bx^3 + O(x^4), cx^2 + dx^3 + O(x^4))$. Then $f(x, h_1(x), h_2(x)) = \varepsilon(-a + k_1 c)x^3 + O(x^4)$ resulting in

$$(Mh)(x) = \begin{pmatrix} -Gax^2 + (-Gb - \varepsilon a + k_1 \varepsilon c)x^3 + O(x^4) \\ k_1 cx^2 + k_1 dx^3 + O(x^4) \end{pmatrix}.$$

Solving for $(Mh)(x) = \overline{0}$ as in (10), we find $h_1(x) = h_2(x) = O(x^4)$. Thus up to third order, $h_1(x) = h_2(x) = 0$, and a center manifold is approximately the $u-$axis. The variable u corresponds to the variable A of the original system.

When $G = 0$, from (21) and (22) it can be seen immediately that the system has a two-dimensional center manifold, corresponding to zero as a double eigenvalue, and a one dimensional stable manifold since the other eigenvalue is $-k_1$. Here $h : V \to \mathbb{R}$, $V \subset \mathbb{R}^2$, a neighborhood of the origin. We let $h(x) = h(x_1, x_2) = ax_1^2 + bx_2^2 + cx_1x_2 + dx_1^3 + ex_2^3 + fx_1^2x_2 + jx_1x_2^2$. Then we calculate

$$\begin{aligned}(Mh)(x) &= \begin{pmatrix} h_{x_1}(x_1, x_2), & h_{x_2}(x_1, x_2) \end{pmatrix} \cdot \begin{pmatrix} k_2x_2 + g(x_1, x_2, h(x_1, x_2)) \\ g(x_1, x_2, h(x_1, x_2)) \end{pmatrix} + k_1 h(x_1, x_2) \\ &= (2ak_2 + ck_1)x_1x_2 + (ak_1)x_1^2 + (ck_2 + bk_1)x_2^2 + (dk_1)x_1^3 + (jk_2 + ek_1)x_2^3 \\ &\quad + (3dk_2 - 2a\varepsilon - c\varepsilon + fk_1)x_1^2x_2 + (2fk_2 - c\varepsilon - 2b\varepsilon + jk_1)x_1x_2^2\end{aligned}$$

resulting in $h(x_1, x_2) = O(x^4)$, thus $h(x_1, x_2) = 0$ up to order three. Hence in this case a center manifold is the $uv-$plane. This corresponds to the M^*A-plane in the original variables.

5.5. **Connections.** Regardless of the value of G, the behavior of the entire system can be observed by considering a center manifold. Here is where comparisons can be made with work done by the food scientists at the SRC. The results obtained by the scientists are based on numerical methods. We can confirm their results through our consideration of the center manifolds we found. In the case $G \neq 0$, they considered a particular set of constant values: $k_1 = 1,\ k_2 = 4,\ k_3 = 100$, and $k_4 = 2$ (thus $G > 0$). The numerical methods show the variable A growing to a certain upper bound, while M and M^* both decay to zero. This agrees with our assessment of the behavior of the system based on the fact that the center manifold is the $A = u-$axis. The behavior of the system is depicted in Figure 1. In this figure, the trajectory either approaches the $u-$axis or grows from it, depending on whether the initial values for v are positive or negative. In the food problem, all values will be nonnegative, thus reflecting the decay to the $u-$axis. Additionally, we can represent the case for $G < 0$, as in Figure 2. In this case, all trajectories are attracted to the $u-$axis, regardless of their initial points. For the case $G = 0$, the behavior of the system was investigated by the scientists for various values of ε and k_1. In all cases they found A to grow, either with or without bound. The variable M^* exhibits different types of behavior: growing with or without bound or decaying to zero. Lastly, M goes to zero, agreeing with our finding of the AM^*-plane being the center manifold.

In both cases for G above, the center manifolds remain invariant under all known Lie symmetries according to Lemma 4.1. This straightforward calculation is included

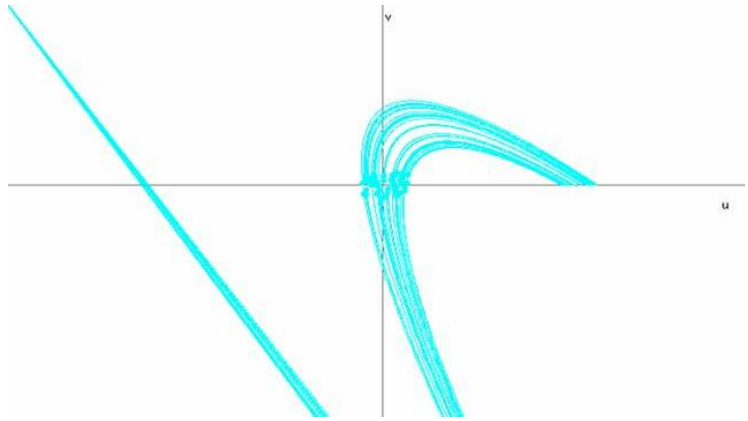

FIGURE 1. Trajectories and the center manifold for $G > 0$ with parameter values $k_1 = 1,\ k_2 = 4,\ k_3 = 100$ and $k_4 = 2$.

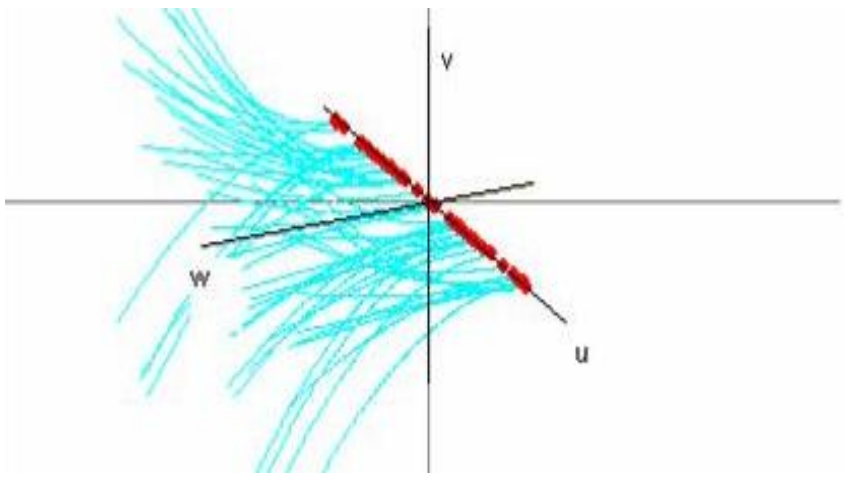

FIGURE 2. Trajectories and the center manifold for $G < 0$ with parameter values $k_1 = 1,\ k_2 = 2,\ k_3 = 100$ and $k_4 = 4$.

in [DP2]. Thus the Lie symmetries cannot provide us with further information about any additional center manifolds. It may be that the ones we found are unique. We would like to say more about how the Lie symmetries act on these manifolds: where do they map certain trajectories on the center manifold, are there any solutions that are invariant under action by the symmetry? These are questions that remain unanswered at this time.

6. Future Work

Some work remains to be done on the analysis of the model of *Staphylococcus aureus* using Lie symmetry and dynamical systems methods. So far, we have found no way to compute the actual Lie symmetries from the infinitesimal generators. This is a difficult problem, actually equivalent to finding the solution of the original equations. We have consulted with Cicogna and Gaeta on this problem and will continue to consider techniques such as exponentiation of the infinitesimal generators in the hopes that we may find an approximation of some of the symmetries. Additionally, there are systems

of equations arising in the research of other scientists at the Natick SRC, such as the equations describing tear propagation in the material used to construct parachutes. We hope to consider the center manifold and symmetry structures of additional systems in future work.

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