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To the Fourth Dimension and Beyond

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The negative signs in the spacetime dot product lead to the famous effects of relativity (time dilation, length contraction, etc.) at speeds close to the speed of light.

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It's anything we want it to be, depending on what we are modeling.

In fact, in pure mathematical terms we can think of four-dimensional geometry, *independent* of any physical model.

The mathematical idea of a fourth dimension predates Einstein by many years–back to the 1840s at least.

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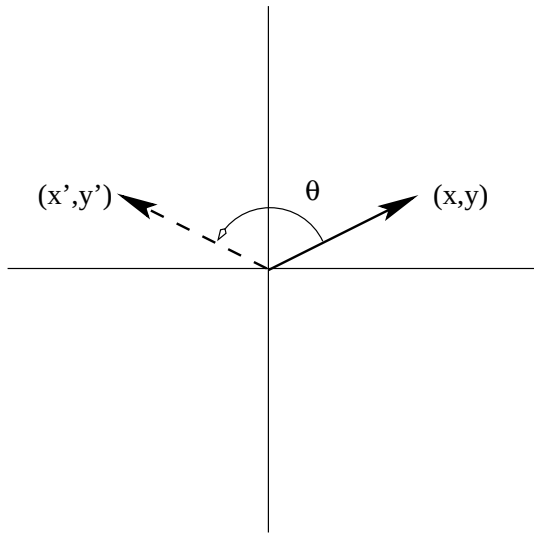

Modern solution: represent (x, y) as a column vector and multiply by an orthogonal matrix:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

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so that the new coordinates are

$$\begin{aligned} x' &= (\cos\theta)x - (\sin\theta)y, \\ y' &= (\sin\theta)x + (\cos\theta)y, \end{aligned}$$

which we can check to be correct by trigonometry.

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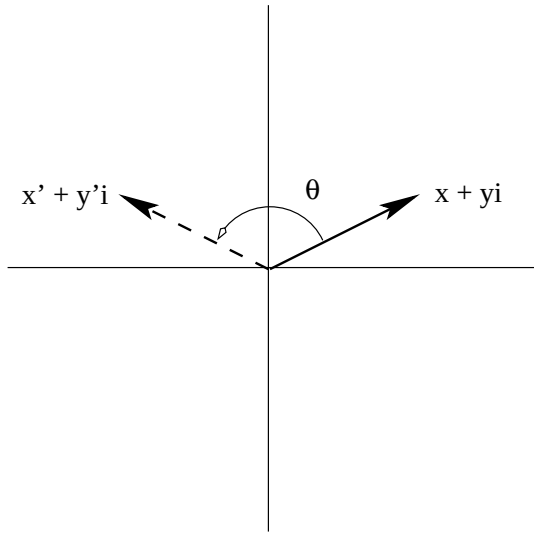

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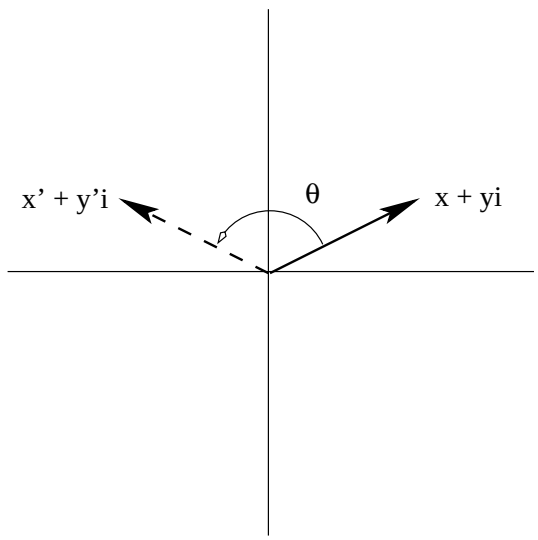

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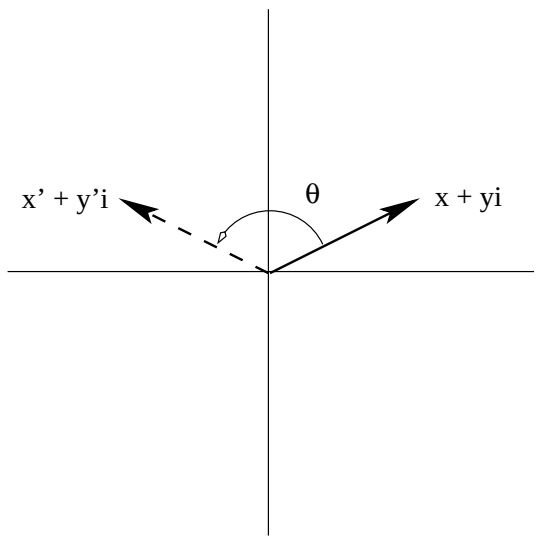

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and multiply $x + y\mathbf{i}$ by the complex number $e^{\mathbf{i}\theta} = \cos\theta + \sin\theta\mathbf{i}$:

$$x' + y'\mathbf{i} = (\cos\theta + \sin\theta\mathbf{i})(x + y\mathbf{i})$$

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- rotations in three dimensional space should be represented by multiplication of hypercomplex numbers

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He defined the *norm* of such a number as

$$|\mathbf{p}| = \sqrt{a^2 + b^2 + c^2}, \tag{1}$$

and in analogy with the complex numbers required that

$$|\mathbf{pq}| = |\mathbf{p}||\mathbf{q}|, \tag{2}$$

for any two triplets $\mathbf{p}$, $\mathbf{q}$.

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$$(a\mathbf{1} + b\mathbf{i} + c\mathbf{j})^2 = (a^2 - b^2 - c^2)\mathbf{1} + 2ab\mathbf{i} + 2ac\mathbf{j} + 2bc\mathbf{ij}$$

using the laws of (complex) addition and multiplication.

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"Behold me therefore tempted for a moment to fancy that $\mathbf{ij} = 0$. *But this seemed odd and uncomfortable...."* – Hamilton to John Graves, 17 Oct 1843.

Hamilton struggled with this for years.

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Thus the troublesome last term would be eliminated if $\mathbf{ji} = -\mathbf{ij}$; that is, if he *discarded the commutative law of multiplication.*

(No one had ever seriously considered this before.)

Hamilton named this product $\mathbf{k} = \mathbf{ij} = -\mathbf{ji}$. Since he was looking for a way to multiply triplets, he expected that

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The breakthrough came on 16 October 1843.

Sometime in the course of his investigations, Hamilton realized that he could think of $\mathbf{k}$ as defining a new spatial direction–he "jumped with $\mathbf{k}$ into a fourth dimension."

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So then what should $\mathbf{k}^2$ be?

On 16 October 1843, Hamilton and his wife were walking to a meeting of the Royal Irish Academy in Dublin and came up to the Brougham Bridge. "...I then and there felt the galvanic circuit of thought closed, and the sparks which fell from it were the fundamental equations between $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$..."

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$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1} \tag{4}$$

$$\begin{aligned} \mathbf{jk} &= \mathbf{i} = -\mathbf{kj} \\ \mathbf{ki} &= \mathbf{j} = -\mathbf{ik} \\ \mathbf{ij} &= \mathbf{k} = -\mathbf{ji} \end{aligned} \tag{5}$$

Hamilton had discovered a set of four-dimensional hypercomplex numbers

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$$\mathbf{q} = \underbrace{q_0\mathbf{1}}_{\text{“scalar”}} + \underbrace{q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}}_{\text{“vector”}}$$

$$= q_0\mathbf{1} + \vec{q} \text{ where } \vec{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}.$$

“Vector is a useless survival, or offshoot from quaternions, and has never been of the slightest use to any creature.” – Lord Kelvin, 1896.

Multiplication of quaternions

$$\mathbf{p} = p_0\mathbf{1} + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
$$\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

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$$\begin{aligned}\mathbf{pq} = &(p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3)\mathbf{1} \\ &+ (p_0q_1 + p_1q_0 + p_2q_3 - p_3q_2)\mathbf{i} \\ &+ (p_0q_2 + p_2q_0 + p_3q_1 - p_1q_3)\mathbf{j} \\ &+ (p_0q_3 + p_3q_0 + p_1q_2 - p_2q_1)\mathbf{k}\end{aligned}$$

–somewhat intimidating at first glance!

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–somewhat intimidating at first glance!
A simpler case–multiplying two pure vectors–gives some insight.

Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
$$\vec{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
$$\vec{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

$$\vec{p}\,\vec{q} = (p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})$$

Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
$$\vec{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

$$\begin{aligned}\vec{p}\,\vec{q} &= (p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) \\ &= -(p_1q_1 + p_2q_2 + p_3q_3)\mathbf{1} + (p_2q_3 - p_3q_2)\mathbf{i} - (p_1q_3 - p_3q_1)\mathbf{j} + (p_1q_2 - p_2q_1)\mathbf{k}\end{aligned}$$

Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
$$\vec{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

$$\begin{aligned}\vec{p}\,\vec{q} &= (p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k})(q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) \\ &= -\underbrace{(p_1q_1 + p_2q_2 + p_3q_3)}_{\vec{p}\cdot\vec{q}}\mathbf{1} + \underbrace{(p_2q_3 - p_3q_2)\mathbf{i} - (p_1q_3 - p_3q_1)\mathbf{j} + (p_1q_2 - p_2q_1)\mathbf{k}}_{\vec{p}\times\vec{q}}\end{aligned}$$

Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
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Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
$$\vec{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

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Multiplying quaternions (again)

$$\begin{aligned}\mathbf{p} &= p_0\mathbf{1} + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}\\ &= p_0\mathbf{1} + \vec{p}\\ \mathbf{q} &= q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}\\ &= q_0\mathbf{1} + \vec{q}\end{aligned}$$

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$$\mathbf{pq} = (p_0\mathbf{1} + \vec{p})(q_0\mathbf{1} + \vec{q})$$

Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
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Multiplying quaternions (again)

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$$\begin{aligned}\mathbf{pq} &= (p_0\mathbf{1} + \vec{p})(q_0\mathbf{1} + \vec{q})\\ &= p_0q_0\mathbf{1} + p_0\vec{q} + q_0\vec{p} + \vec{p}\vec{q}\end{aligned}$$

Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
$$\vec{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

$$\vec{p}\,\vec{q} = -(\vec{p}\cdot\vec{q})\mathbf{1} + \vec{p}\times\vec{q}$$

Multiplying quaternions (again)

$$\begin{aligned}\mathbf{p} &= p_0\mathbf{1} + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}\\ &= p_0\mathbf{1} + \vec{p}\\ \mathbf{q} &= q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}\\ &= q_0\mathbf{1} + \vec{q}\end{aligned}$$

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Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
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$$\vec{p}\,\vec{q} = -(\vec{p}\cdot\vec{q})\mathbf{1} + \vec{p}\times\vec{q}$$

Multiplying quaternions (again)

$$\begin{aligned}\mathbf{p} &= p_0\mathbf{1} + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}\\ &= p_0\mathbf{1} + \vec{p}\\ \mathbf{q} &= q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}\\ &= q_0\mathbf{1} + \vec{q}\end{aligned}$$

$$\begin{aligned}\mathbf{pq} &= (p_0\mathbf{1} + \vec{p})(q_0\mathbf{1} + \vec{q})\\ &= p_0q_0\mathbf{1} + p_0\vec{q} + q_0\vec{p} + \vec{p}\vec{q}\\ &= p_0q_0\mathbf{1} + p_0\vec{q} + q_0\vec{p} + (-\vec{p}\cdot\vec{q})\mathbf{1} + \vec{p}\times\vec{q}\\ &= (p_0q_0 - \vec{p}\cdot\vec{q})\mathbf{1} + p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q}\end{aligned}$$

Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
$$\vec{q} = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

$$\vec{p}\,\vec{q} = -(\vec{p}\cdot\vec{q})\mathbf{1} + \vec{p}\times\vec{q}$$

Multiplying quaternions (again)

$$\begin{aligned}
\mathbf{p} &= p_0\mathbf{1} + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} \\
&= p_0\mathbf{1} + \vec{p} \\
\mathbf{q} &= q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \\
&= q_0\mathbf{1} + \vec{q}
\end{aligned}$$

$$\begin{aligned}
\mathbf{pq} &= (p_0\mathbf{1} + \vec{p})(q_0\mathbf{1} + \vec{q}) \\
&= p_0q_0\mathbf{1} + p_0\vec{q} + q_0\vec{p} + \vec{p}\vec{q} \\
&= p_0q_0\mathbf{1} + p_0\vec{q} + q_0\vec{p} + (-\vec{p}\cdot\vec{q})\mathbf{1} + \vec{p}\times\vec{q} \\
&= (p_0q_0 - \vec{p}\cdot\vec{q})\mathbf{1} + p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q} \\
&= (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3)\mathbf{1} + p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q}
\end{aligned}$$

Multiplying vectors

$$\vec{p} = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$$
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$$\mathbf{pq} = (p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3)\mathbf{1} + p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q}$$

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$$\mathbf{pq} = \underbrace{(p_0q_0 - p_1q_1 - p_2q_2 - p_3q_3)}_{\text{Einstein's dot product}}\mathbf{1} + p_0\vec{q} + q_0\vec{p} + \vec{p}\times\vec{q}$$

Definition. *The* norm *of a quaternion* $\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ *is*

$$|\mathbf{q}| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{6}$$

Definition. *The* conjugate *of a quaternion* $\mathbf{q} = q_0\mathbf{1}+q_1\mathbf{i}+q_2\mathbf{j}+q_3\mathbf{k}$ *is*

$$\bar{\mathbf{q}} = q_0\mathbf{1} - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}. \tag{7}$$

Definition. *The* norm *of a quaternion* $\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ *is*

$$|\mathbf{q}| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{8}$$

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$$\bar{\mathbf{q}} = q_0\mathbf{1} - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}. \tag{9}$$

Note that

$$\begin{aligned}\mathbf{q}\bar{\mathbf{q}} &= (q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(q_0\mathbf{1} - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}) \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2\end{aligned}$$

Definition. *The* norm *of a quaternion* $\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ *is*

$$|\mathbf{q}| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}. \tag{10}$$

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$$\bar{\mathbf{q}} = q_0\mathbf{1} - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}. \tag{11}$$

Note that

$$\begin{aligned}
\mathbf{q}\bar{\mathbf{q}} &= (q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(q_0\mathbf{1} - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}) \\
&= q_0^2 + q_1^2 + q_2^2 + q_3^2 \\
&= |\mathbf{q}|^2
\end{aligned}$$

Definition. *The* norm *of a quaternion* $\mathbf{q} = q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$ *is*

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$$\bar{\mathbf{q}} = q_0\mathbf{1} - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}. \tag{13}$$

Note that

$$\begin{aligned}\mathbf{q}\bar{\mathbf{q}} &= (q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})(q_0\mathbf{1} - q_1\mathbf{i} - q_2\mathbf{j} - q_3\mathbf{k}) \\ &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \\ &= |\mathbf{q}|^2\end{aligned}$$

so every $\mathbf{q} \neq 0$ has a multiplicative inverse

$$\mathbf{q}^{-1} = \frac{1}{|\mathbf{q}|^2}\bar{\mathbf{q}}.$$

Hamilton wanted to find an extension of the complex numbers that would act on three-dimensional space the way complex numbers do on two dimensions. Did he succeed?

Recall that Hamilton wanted his new numbers to have the following properties:

- every nonzero number $\mathbf{q}$ should have an inverse $\mathbf{q}^{-1}$;
- the *norm* (i.e. magnitude) should follow the product rule

  $$|\mathbf{pq}| = |\mathbf{p}||\mathbf{q}|$$

  for all numbers $\mathbf{p}$ and $\mathbf{q}$.
- rotations in three dimensional space can be represented by multiplication of hypercomplex numbers

We just showed that every nonzero quaternion $\mathbf{q} \neq 0$ has a multiplicative inverse

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A little algebra confirms that

$$|\mathbf{pq}| = |\mathbf{p}||\mathbf{q}|$$

for any two quaternions $\mathbf{p}, \mathbf{q}$.

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What about rotations?

Every rotation in three-dimensional space is a rotation about an *axis* defined by a unit vector $\vec{u}$ (Euler's theorem).

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A year after discovering quaternions, Hamilton found the formula for rotating a vector $\vec{v}$ about an axis defined by a unit vector $\vec{u}$.

To rotate $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ counterclockwise about $\vec{u}$ by an angle θ, define the quaternion

$$\mathbf{q} = \cos\frac{\theta}{2}\mathbf{1} + \sin\frac{\theta}{2}\vec{u}$$

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To rotate $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ counterclockwise about $\vec{u}$ by an angle θ, define the quaternion

$$\mathbf{q} = \cos\frac{\theta}{2}\mathbf{1} + \sin\frac{\theta}{2}\vec{u}$$

and form the product

$$\vec{v'} = \mathbf{q}\,\vec{v}\,\mathbf{q}^{-1}$$

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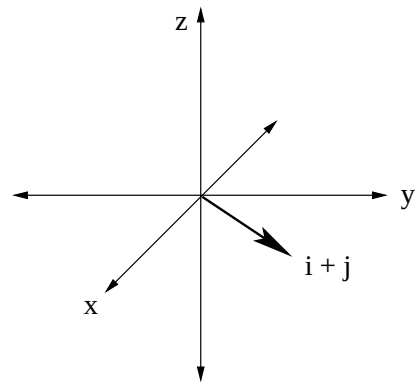
and form the product

$$\begin{aligned}\vec{v'} &= \mathbf{q}\,\vec{v}\,\mathbf{q}^{-1} \\ &= (\cos\frac{\theta}{2}\mathbf{1} + \sin\frac{\theta}{2}\vec{u})\,\vec{v}\,(\cos\frac{\theta}{2}\mathbf{1} - \sin\frac{\theta}{2}\vec{u})\end{aligned}$$

Example: rotate $\vec{v} = \mathbf{i} + \mathbf{j}$ about the positive y-axis counter-clockwise by $\pi/3$ radians.



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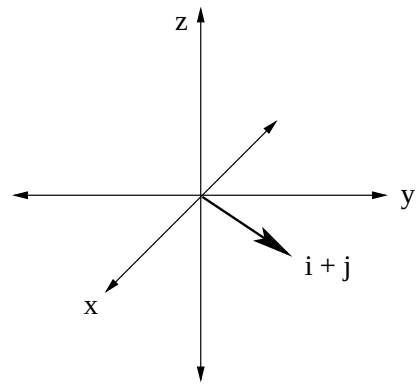

Matrix solution:

$$\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \cos\frac{\pi}{3} & 0 & \sin\frac{\pi}{3} \\ 0 & 1 & 0 \\ -\sin\frac{\pi}{3} & 0 & \cos\frac{\pi}{3} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

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$$= \frac{1}{2}\mathbf{i} + \mathbf{j} - \frac{\sqrt{3}}{2}\mathbf{k}$$

Example: rotate $\vec{v} = \mathbf{i} + \mathbf{j}$ about the positive y-axis counter-clockwise by $\pi/3$ radians.

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- Thus $\mathbf{q}$ and $-\mathbf{q}$ correspond to the same rotation matrix.

There are two quaternions for every rotation matrix.

However, an additional rotation of 4π results in the *same* rotation quaternion:

$$\cos\frac{(\theta+4\pi)}{2}\mathbf{1}+\sin\frac{(\theta+4\pi)}{2}\vec{u}=\cos(\frac{\theta}{2}+2\pi)\mathbf{1}+\sin(\frac{\theta}{2}+2\pi)\vec{u}$$

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Thus rotations of 360 degrees (2π radians) are somehow different in the quaternion world than rotations of 720 degrees (4π radians).

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Which picture is *physically* correct?

If there is no physical difference between rotations of 360 degrees and rotations of 720 degrees, the matrix formulation of rotations is correct.

In order for the quaternion picture to be correct, there would have to be some detectable difference between a 360 degree rotation and a 720 degree rotation.

The coffee cup trick

The sphere and the hypersphere

In three-dimensional space, the set of vectors defined by

$$S^2 = \{p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} \,|\, p_1^2 + p_2^2 + p_3^2 = 1\}$$

is the 2-dimensional *unit sphere* centered at the origin.

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In the four-dimensional space of quaternions, the set of unit quaternions

$$S^3 = \{q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \,|\, q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\}$$

is called the *hypersphere.*

The sphere and the hypersphere

It's easy to show that every rotation quaternion $\mathbf{q} = \cos\frac{\theta}{2}\mathbf{1} + \sin\frac{\theta}{2}\vec{u}$ is a unit quaternion and vice versa. Thus,

$$S^3 = \{q_0\mathbf{1} + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \,|\, q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\}$$

$$= \{\cos\frac{\theta}{2}\mathbf{1} + \sin\frac{\theta}{2}\vec{u} \,|\, 0 \leq \theta \leq 4\pi,\ \vec{u} \text{ unit vector}\}$$

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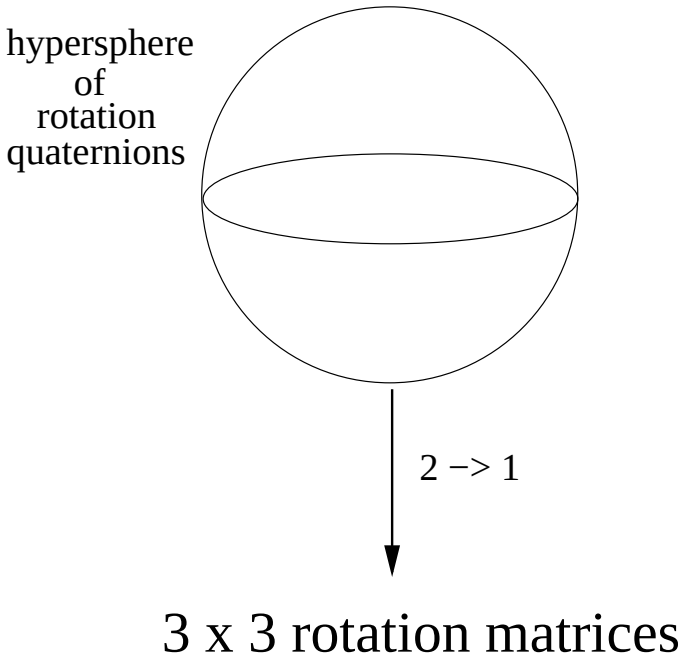
Thus the set of rotation quaternions is the hypersphere.



The sphere and the hypersphere

Every unit quaternion $\mathbf{q} = \cos\frac{\theta}{2}\mathbf{1} + \sin\frac{\theta}{2}\vec{u}$ in the hypersphere projects to its unit vector component $\vec{u}$ in the unit sphere.

The set of unit quaternions corresponding to the equator of the unit sphere is a *torus* embedded in four-dimensional space. What does it look like?

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  - efficient use of computer memory
- **Quantum mechanics:** the unitary operators for a spin 1/2 particle are unit quaternions
- **Computer graphics / game engines:** for the same reasons as NASA, plus:
  - smoother motion interpolation in rotation of objects
  - avoids distortion (shear) due to roundoff error

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- Einstein's general theory of relativity is *very* good for describing and predicting the effects of gravity.
- Quantum mechanics is *amazingly* good at describing and predicting the effects of the other three forces in nature.

However – these two theories are mathematically incompatible!

Finding a theory that unifies them is *the* outstanding problem in physics. Some physicists believe the answer involves even higher dimensions.

Higher dimensions and string theory

In what has been called "string theory" (or *M-theory*), elementary particles are represented by "strings" which move in a space of 10 (or perhaps 11) dimensions.

At every point in 4-dimensional spacetime, string theory postulates an additional 6-dimensional space called a *Calabi-Yau manifold*.

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At every point in 4-dimensional spacetime, string theory postulates an additional 6-dimensional space called a *Calabi-Yau manifold*.

These extra dimensions are too small to be detected by human senses, or by any technical means we now possess.

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- The extra dimensions, while small, give enough extra room for effects that would reconcile relativity and quantum mechanics
- Gravity is a natural consequence of the field equations
- They let people make cool science programs for PBS

Looking back and ahead

Edward Witten (IAS) once said, "String theory is a piece of 21st century physics that somehow dropped into the 20th century."

Quaternions are a piece of 20th century mathematics that dropped into the 19th century, thanks to the genius of Hamilton.

The problems that quaternions can solve weren't foreseen in the 19th century... we don't know yet what the 10-dimensional mathematics of string theory may lead to.