EXPLORING THE NUMERICAL RANGES

Roger Lautzenheiser

Rose-Hulman Institute of Technology

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Let T be a $n \times n$ matrix with complex entries.

$$W(T) = \{(Tx, x) : x \in C^n \text{ and } \|x\| = 1\}$$

Recall that for $x, y \in C^n$, $\qquad (x, y) = \sum x_i \overline{y_i}$

Also note that the dot product can be written as $(Tx, x) = x^*Tx$ using usual matrix multiplication.

HISTORY

Otto Toeplitz (1918) introduced $W(T)$ (called it the field of values) and proved that the outer boundary of $W(T)$ is CONVEX.

Felix Hausdorff (1919) proved that $W(T)$ is CONVEX.

The fact that $W(T)$ is a CONVEX, COMPACT subset of C is referred to as the Toeplitz-Hausdorff Theorem.

Applications include

- Analysis of Iterative Methods - Eiermann
- Analysis of ADI Method - Starke
- Dynamical Systems, Lyapunov's Thm - Johnson
- Application in NMR-Spectroscopy - Helmke,...

EXAMPLES

$$T = \text{ zero matrix } \Longrightarrow W(T) = \{0\}$$

$$T = I \Longrightarrow W(T) = \{1\}$$

$$T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \Longrightarrow W(T) = [2, 3]$$

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow W(T) = \{z : |z| \leq 0.5\}$$

$$x = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{pmatrix} \text{ with } r_1^2 + r_2^2 = 1$$

$$\left(\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{pmatrix}, \begin{pmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{pmatrix} \right) = 2\ r_1^2 + 3\ r_2^2$$

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{pmatrix}, \begin{pmatrix} r_1 e^{i\theta_1} \\ r_2 e^{i\theta_2} \end{pmatrix} \right) = r_1 r_2 e^{i(\theta_2 - \theta_1)}$$

The max of $r_1 r_2$ subject to $r_1^2 + r_2^2 = 1$ is $\frac{1}{2}$.

PROPERTIES of $W(T)$

- $W(\alpha I) = \{\alpha\}$
- $W(\alpha T) = \alpha W(T)$
- $W(T + \alpha I) = W(T) + \alpha$
- $W(T) \subseteq \{z : |z| \leq \|T\|\}$
- T has real entries $\Longrightarrow W(T) = \overline{W(T)}$
- $W(D) =$ convex hull of diagonal entries.

- T is unitarily equivalent to $S \Longrightarrow W(T) = W(S)$
- T is Hermitian $\Longrightarrow W(T) = [\lambda_{min}, \lambda_{max}]$

  ($T = T^*$ or $T = UDU^*$ with D diagonal and real)
- T is Normal $\Longrightarrow W(T) =$ convex hull of $\sigma(T)$

  ($TT^* = T^*T$ or $T = UDU^*$ with D diagonal and complex, unless Hermitian)
- $\lambda \in \sigma(T) \Longrightarrow \lambda \in W(T)$
- T is similar to $S \Longrightarrow W(T)$, $W(S)$ not necessarily the same

CONVERSELY (contrast with corresponding properties of eigenvalues)

- $W(T) = \{0\} \Longrightarrow T =$ the zero matrix

  Proof: Be wise, polarize

- $W(T) = [\mu, \nu] \Longrightarrow T = T^*$

  Proof: $((T - T^*)x, x) = (Tx, x) - (T^*x, x)$

  $= (Tx, x) - (x, Tx) = (Tx, x) - \overline{(Tx, x)} = 0$

- $W(T) =$ any line segment, then $T = \alpha + \beta S$

  where $S = S^*$, so T is normal.

Convexity when T is 2×2

- Can assume the matrix is $\begin{pmatrix} \lambda & \alpha \\ 0 & \mu \end{pmatrix}$

- $\begin{pmatrix} \lambda & \alpha \\ 0 & \mu \end{pmatrix} - \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda - \mu & \alpha \\ 0 & 0 \end{pmatrix}$

  So easy if $\alpha = 0$ or $\lambda = \mu$

- Write $\begin{pmatrix} \lambda-\mu & \alpha \\ 0 & 0 \end{pmatrix} = (\lambda-\mu)\begin{pmatrix} 1 & \frac{\alpha}{\lambda-\mu} \\ 0 & 0 \end{pmatrix}$

- $\begin{pmatrix} 1 & \frac{\alpha}{\lambda-\mu} \\ 0 & 0 \end{pmatrix} \approx \begin{pmatrix} 1 & \left|\frac{\alpha}{\lambda-\mu}\right| \\ 0 & 0 \end{pmatrix}$, That is,

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix}\begin{pmatrix} 1 & |z|e^{i\theta} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \begin{pmatrix} 1 & |z| \\ 0 & 0 \end{pmatrix}$$

Summary of the 2×2 Case

For $T = \begin{pmatrix} \lambda & \alpha \\ 0 & \mu \end{pmatrix}$

$W(T)$ is an elliptical disk (possibly degenerate)

Foci: eigenvalues of T;   Center: $\frac{tr(A)}{2}$

Length of Major Axis: $(|\lambda - \mu|^2 + |\alpha|^2)^{\frac{1}{2}}$

Length of Minor Axis: $|\alpha|$

Convexity in General

- Note: For P a projection, $W(PTP) \subseteq W(T)$.

Proof: $x \in \mathcal{R}(P) \Rightarrow (PTPx, x) = (Tx, Px) = (Tx, x)$

- Let $\lambda \neq \mu$ be in $W(T)$. So $\lambda = (Tx, x)$ and

$\mu = (Ty, y)$ for lin indep unit vectors x and y.

- Let P be the projection onto $span\{x, y\}$.

- $W(T)$ is convex since $W(PTP)$ is an elliptical disk or a line segment containing λ and μ.

3×3 Case

- Kippenhahn (1951) gave characterization in terms of algebraic curves

- REU project (1997) at William & Mary resulted in a paper which translates Kippenhahn's results into more understandable statements.

- Possible shapes

  - convex hull of the eigenvalues
  - ellipse or cone-like shape
  - boundary contains a flat portion
  - oval shape

Geometry of $\partial(W(T))$

- Let λ be a corner point and $(Tx, x) = \lambda$.

- If $Tx \neq \lambda x$, then Tx and x are lin indep.

- Let P be the projection onto $\{x, Tx\}$.

- Obtain a contradiction knowing that $W(PTP)$ is elliptical.

Furthermore, λ is a reducing eigenvalue. This means that T can be written as a direct sum $\lambda I \oplus T'$.

Reason: Applying Schur's Lemma, T is unitarily equivalent to

$$\begin{pmatrix} \lambda & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & 0 & * & \cdots & \vdots \\ 0 & \cdots & 0 & \ddots & * \\ 0 & 0 & \cdots & 0 & * \end{pmatrix}$$

What if $W(T) =$ is a convex polygon?

- If $n \leq 4$, then T is normal.

- Otherwise, $T = N \oplus T'$ with N normal and $W(T') \subseteq W(N)$

General Result on Direct Sums

Let $T = A \oplus B$ then,

$W(T) =$ the convex hull of $W(A)$ and $W(B)$.

Let $T = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ and $x = \begin{pmatrix} y \\ z \end{pmatrix}$

$$\begin{aligned}(Tx, x) &= (Ay, y) + (Bz, z) \\ &= \|y\|^2 \left(A\tfrac{y}{\|y\|}, \tfrac{y}{\|y\|}\right) + \|z\|^2 \left(B\tfrac{z}{\|z\|}, \tfrac{z}{\|z\|}\right)\end{aligned}$$

Generalizations

- $W_k(T) =$ set of $tr(PTP)$ with $rank(P) = k$
- $W_c(T) =$ set of $\sum_{i=1}^{n} c_i(Tv_i, v_i)$ with $\{v_i\}$ o.n.
- $W_C(T) =$ set of $tr(CU^*AU)$ where U is unitary
- $W(T_1, T_2) = \{((T_1x, x), (T_2x, x)) : x \in C^n \text{ and } \|x\| = 1\}$
- Many others...

CLASS RESEARCH PROJECT

Let T be a $n \times n$ matrix with complex entries, and define the numerical range as

$$W(T) = \{(Tx, x) : x \in C^n \text{ and } \|x\| = 1\}$$

$W(T)$ is a set of complex numbers associated with T.
Question: How is the geometry of $W(T)$ related to T?

The idea is to ask and answer as many questions about this relationship as possible.

Devote Fridays to presentations.

Mechanics of Grading - Project counts 1/3 of your grade

- (1 pt) Non-silly (trivial ok) conjecture
- (1 pt) Supporting evidence for a conjecture
- (1 pt) An example
- (2 pts) Proof of a conjecture
- (2 pts) Counterexample of a conjecture
- (3 pts) Theorem and proof

(*) 1. [RGL, example] If $T = I$, then $W(T)$=$\{1\}$. Proof: if $x \in C^n$ of length 1, then
$(Tx, x) = (Ix, x) = (x, x) = \|x\|^2 = 1.$

() 2. [RGL, conjecture] If $T = \lambda I$, then $W(T) = \lambda$.

(*) 3. [RGL, conjecture] If the entries of T are real, then $W(T) \subset R$.

(*) 4. [conjecture 3 - Jeff] False. Let $T = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

and $x = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 1 \end{pmatrix}$

(*) 5. [Thm/proof, Matt] The diagonal elements of T are in $W(T)$. Proof: Let $x = e_i$ to show that the i^{th} diagonal element is in $W(T)$.

(*) 6. [Thm/proof, Kevin] Each eigenvalue of T is in $W(T)$.

(*) 7. [Michael, Chad and Brian, example] If $T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then $W(T) = [1, 2]$

() 8. [Jon, conjecture] If T is diagonal, then $W(T)$ consists of the diagonal elements and the stuff inside.

(*) 9.[Jon,John, and Rick, example] If $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $W(T)$ is the closed disk with center 0 and radius $\frac{1}{2}$.

(*) 10.[Matt, Theorem/proof] If T and S are unitarily equivalent, then $W(T) = W(S)$. Proof...

It's never too long until someone conjectures that the numerical range of a 2×2 matrix is an ellipse along with its interior. However, this conjecture has never been proved completely in a class.

References

Horn and Johnson - Topics in Matrix Analysis (1991)

Gustafson and Rao - Numerical Range (1996)

Chi-Kwong Li's website at William & Mary

Papers of Charles Johnson.

Search the web.

Papers cited in MathSciNet

- Before 1960: 5
- $[1960, 1969]$: 107
- $[1970, 1979]$: 413
- $[1980, 1989]$: 512
- $[1990, 1999]$: 650
- After 2000: 264