**Project 2 -- Centers of Mass**[1]

"Many structures and mechanical systems behave as if their masses were concentrated at a single point called the *center of mass*. It is important to know how to locate this point, and it turns out that doing so is basically a mathematical undertaking."[2] In this project you will empirically determine the center of mass for plexiglas cut-outs of three familiar mathematical functions. You'll then use calculus to find the center of mass exactly and check the accuracy of your experimental measurements.

As the plexiglas cut-outs make their way around to your group, fill in the table below. You should express the center of mass as an $x$-coordinate and a $y$-coordinate for each cut-out. (We use the notation $\bar{x}$ and $\bar{y}$ to represent these coordinates.) One way to do the measurement is to balance the cut-out on a rounded pencil eraser and them mark the opposite surface with a washable marker. Use the meter sticks in the classroom to make your measurements.

All of the bounding curves were plotted with a domain of [ 0, 1 ], printed on a laser printer and then cut out of plexiglas sheeting. You will need to convert your physical measurements of $\bar{x}$ and $\bar{y}$ to fit a [ 0, 1 ] scale on the $x$-axis (we'll use a simple proportion). Do this by dividing your measured values for $\bar{x}$ and $\bar{y}$ by the maximum "$x$" length of your cut-out (the green edge of your cut-out goes along the $x$-axis).

| Bounding Curve | Measured $\bar{x}$ | Measured $\bar{y}$ | Scaled $\bar{x}$ | Scaled $\bar{y}$ |
|---|---|---|---|---|
| A. | | | | |
| B. | | | | |
| C. | | | | |

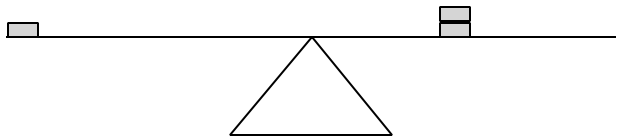
Now you have your empirical data -- let's move toward our goal of making the exact calculations for $\bar{x}$ and $\bar{y}$ using calculus.

If we want to be able to find the exact values for $\bar{x}$ and $\bar{y}$ we must be able to talk the language of balanced systems. Do some balancing experiments with the meter sticks in your classroom and pennies provided by your instructor. Can you write a rule which explains the conditions for the stick to balance? Can you predict the balancing point? Write your ideas in the box below.

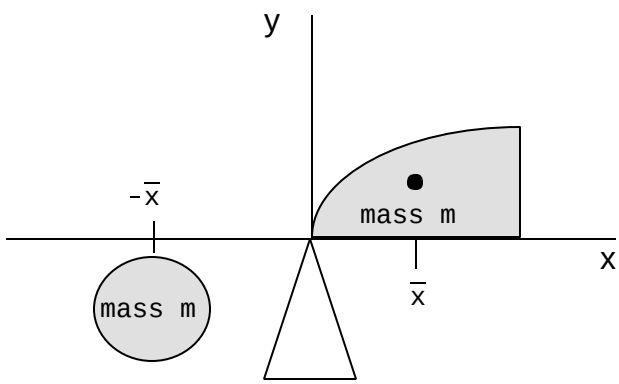
[1]Adapted from Project CALC with permission of the authors, David A. Smith and Lawrence C. Moore.

[2]Calculus, Finney/Thomas, p.425

The contribution of each mass on the balance beam is called a **moment**; when working with a coordinate system, it is convenient to let the *signs* of coordinates do the work of keeping track of "left" and "right." (the easiest coordinate system we could apply to the system shown below is to put the origin at the fulcrum.) We represent the moment of a point mass by the **product** of mass, $m$, and its *directed* distance, $\bar{x}$, from the balance axis (the tip of the fulcrum). The rule for balancing the meter stick is that **the sum of the moments is zero.** (In symbols: $\sum M = 0$)



Imagine that we place one of our cut-outs on the right-hand side of a balance beam, as in the figure below. To achieve balance by placing the same mass on the left at a single point, that point would have to be $\bar{x}$ units away from the origin. In the case of the figure below we speak of *moments with respect to the y-axis* since we are balancing on opposite sides of the line $x = 0$ (i.e., the $y$-axis).




For the figure above, the moment of the cut out with respect to the y-axis is unknown, so we just give it a name: $M_y$. The moment of the point mass on the left is $-m\bar{x}$. The balancing equation is $M_y - m\bar{x} = 0$.

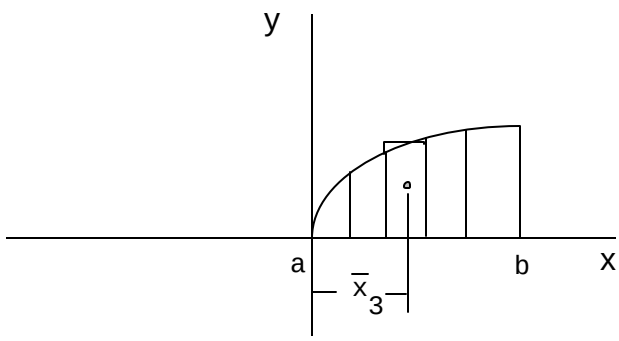
Solve this equation for $\bar{x}$ and write your answer in the box at the right. The problem of finding $\bar{x}$ is now reduced to that of finding two other quantities, $M_y$ and $m$.

$\bar{x} =$

Finding the Mass: Each of your cut-outs has a constant thickness, $q$ , and a constant density, $d$ . Explain briefly why the problem of finding $m$ can be reduced to the problem of finding **area** once you know $q$ and $d$ (Hint: Recall that density is mass per unit volume). Calculate the area and mass of each cutout and record your answers in the table below.

| Cut -Out | Area | Mass |
|---|---|---|
| A. | | |
| B. | | |
| C. | | |

Finding the Moment of a cut-out: We will address this problem using a strategy similar to the one that gave us the idea for finding the area "under a curve". Let's partition our cut-out into $n$ vertical strips and write a moment equation for one the strips (say, the third from the left). Each strip has a width of $D\,x$ .




Recall that:

Moment = (length of the moment arm) x (mass of the object).

Then $M_3 = (\bar{x}_3)(\text{mass of the third partition})$, or

$$M_3 = (\bar{x}_3)(qd \times \text{area}).$$

The area of our strip can be estimated as the area of a rectangle using a mid-point estimation of the rectangle's height: $\text{area} \approx f(\bar{x}_3) \cdot \Delta x$

Then $M_3 = (\bar{x}_3)[qd \cdot f(\bar{x}_3) \cdot \Delta x]$, or rewritten,

$$M_3 = qd \cdot \bar{x}_3 \cdot f(\bar{x}_3) \cdot \Delta x.$$

Similarly, $M_4 = qd \cdot \bar{x}_4 \cdot f(\bar{x}_4) \cdot \Delta x$.

In general, $M_i = qd \cdot \bar{x}_i \cdot f(\bar{x}_i) \cdot \Delta x$.

The total moment, $M_T$, for the cut-out is approximated by the summation of all the strip moments, which is a Riemann Sum:

$$M_T \approx qd \cdot \sum_{i=1}^{n} \bar{x}_i f(\bar{x}_i) \Delta x .$$

If we take the limit of our Riemann Sum as the width of the partition approaches zero, we obtain the integral

$$M_T = M_y = qd \int_a^b x \cdot f(x) dx ,$$

which is the exact moment of our cut-out about the *y*-axis.

Go back to the equation you wrote for $\bar{x}$ and update it knowing what you now know about the mass, *m*, and the moment, $\underline{M_y}$. What happens to $q$ and $d$ ?

If you are ready to move on, it is time to think about how to calculate $\bar{y}$. For that purpose, we need to think about balancing about the *x*-axis -- you can choose whether to mentally rotate your cut-out 90°, physically rotate it 90°, or imagine gravity acting sideways. Convince yourself that the moment of the cut-out with respect to the *x*-axis must satisfy the condition $M_x - m\bar{y} = 0$. Find a formula for $\bar{y}$ in terms of $m$ and $M_x$: _____. The problem of finding $\bar{y}$ is now reduced to that of finding two other quantities, one of which you already know.

You have already done most of the hard work leading to a formula for $M_x$. But be careful -- for $M_y$, the balance axis was parallel to the strips, and for $M_x$ it is perpendicular. (You could also think about using horizontal strips to calculate $M_x$.)

When you have found the exact $\bar{x}$ and $\bar{y}$ for each of your curves, create a table which shows the results of your experiments. Then you are ready to write your project report. Be sure to comment on the accuracy of your empirical data, how you decided to calculate $M_x$, and whether or not you used Derive™ software to calculate your integrals. Include this handout as an appendix to your project report.