Undergraduate Mathematics for the Future: Modeling and Solving Problems, Understanding the New Sciences

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Undergraduate Mathematics

The advancement and perfection of mathematics are intimately connected with the prosperity of the State. -- Napoleon

Throughout history, mathematics has been very helpful to the civilized world. It is a powerful tool, and there are many examples where mathematics has contributed significantly to the advancement of society. While this establishes a strong connection between mathematics and progress, we ask, "why require mathematics for our undergraduates?" We discuss that question through a combined disciplinary and historical perspective of undergraduate mathematics education in America.

The whole of mathematics consists in the organization of a series of aids to the imagination in the process of reasoning. -- Alfred North Whitehead

Mathematics, as a tool for humans, has always been useful. Benjamin Franklin highlighted this in his 1738 paper entitled "On the Usefulness of Mathematics" when he wrote: "What science can be more noble, more excellent, more useful for men, more admirably high and demonstrative, than that of mathematics?" In Franklin's day, mathematics was frequently taught as an art. The art of mathematics exercised the mind in reasoning, memorizing, analyzing patterns, and reciting formulas, proofs, and theorems. There is great beauty in this art of mathematics. Of course, the practical, professional side of mathematics also empowered business people, farmers, surveyors, and navigators of colonial America. Some of these practical skills were found in academic courses. It was important for the growth of our nation that the college-level schools of early America attempted to teach mathematics both as an art and profession.

The chief forms of beauty are order and symmetry and precision which the mathematical sciences demonstrate in a special degree. -- Aristotle

In the early 19th century, undergraduate mathematics began to be taught as the language of science. This kind of mathematics has structure, process, and utility. New subjects in mathematics and science were added to school's curricula, teaching styles were refined (problems were solved and analyzed, conceptual understanding was emphasized), and soon college graduates who knew the language of science were being produced. These graduates could assemble mathematics to solve problems and some used their mathematics to design and build better roads, buildings, bridges, canals, and railroads. American colleges developed curricula that taught mathematics both as an art and as the language of science. America began to produce its own highly skilled college graduates, many becoming engineers and technologists. Mathematics in college was important to empower graduates to become productive, successful citizens.

Mathematics … represents the ultimate structure and order. I associate doing mathematics with control. -- Paul Erdos

By the beginning of the 20th century, high school and college student had the opportunity to learn mathematics as a language of science and utilize it to solve problems. Society had advanced in its use of technology. What part would mathematics play in the new era? Fortunately a new role for undergraduate mathematics was available. In addition to being an art and a language, college-level mathematics had become a science -- the science of measurement. By the early 20th century, the science component of mathematics was available to undergraduates. Many people's jobs required them to perform quantitative measurements. Our society had embraced technology. New devices like cars, planes, radios, televisions, and telephones were developed and refined using the science of mathematics and almost all citizens, especially college graduates, used technology in many ways in their personal and professional lives. College graduates trained as managers were expected to understand, design, and optimize sophisticated plans and operations; use and maintain sophisticated mechanical equipment and transportation systems; design and implement efficient schedules; and understand entirely new technological devices. Mathematics requirements for undergraduates had definitely changed to highlight understanding and use of mathematics as a science.

Computers can never eliminate the need for problem-solving through human ingenuity and intelligence. -- Paul Brock

What challenges does the 21st century and the dawning of the information age bring to undergraduate mathematics? Modern technologies in the form of computers are tremendous tools for visualization and problem solving; therefore, the art, language, and science roles of mathematics have changed dramatically. Technological tools actually perform much of what was considered undergraduate mathematics in the not-so-distant past. People now face new technological and quantitative challenges. College-educated managers and professionals are required to process data and synthesize information, use and understand information technology, optimize elaborate plans, confront complexity, and leverage new technologies. The diverse missions of today's businesses and industries require people with a multitude of skills to confront the myriad of challenges of the modern world. Many schools' educational goals take these new challenges into account. Today's college core mathematics programs need to serve the role to produce creative, confident, competent problem solvers. An essential component of modern undergraduate mathematics becomes modeling (forming and analyzing problems, using technical tools, and implementing solutions) with an emphasis on interdisciplinary problem solving. College graduates will need to use technological tools to solve problems from every facet of life (physical sciences, life sciences, social sciences, behavioral sciences, political sciences, technology, and humanities). They need to become quantitative and interdisciplinary problem-solvers to serve society and satisfy the world's diverse needs.

If science is viewed as an industrial establishment, then mathematics is an associated power plant which feeds a certain kind of indispensable energy into the establishment. -- Solomon Bockner

Over these past two centuries the needs of the society have changed. Fortunately, during these changes undergraduate mathematics has also changed. This reinventing of undergraduate

mathematics has kept undergraduate mathematics successfully serving the needs of society. College graduates, in filling their role as citizens, business leaders and productive professionals, provide the intellectual and technical leadership of society and are required to be professionally competent. More than just knowing one's job or establishing one's proficiency, competence in the 21st century includes having the attributes of creativity and confidence and skills in problem solving. Mathematics, in its forms of an art, a language, a science, and a problem-solving tool, contribute greatly to the development of these essential attributes. Our students need to study mathematics because of its importance in the everyday world. Many daily situations bring people into contact with mathematics, including buying products, conduction business, producing products, managing people and technology, using science and technology. These are the reasons we require our nation's undergraduates to learn core-level mathematics.

But of all the sciences cultivated by mankind, none are more useful than mathematics, to call forth a spirit of enterprise and enquiry. -- Consider Sterry

What is core mathematics?

What is the core mathematics program as we enter the 21st century and the information age? What do we teach and how do we teach it?

Hence we must believe that all the sciences are so interconnected, that it is much easier to study them all together than to isolate one from all the others. If, therefore, anyone wishes to search out the truth of things in serious earnest, he ought not select one special science, for all the sciences are cojoined with each other and interdependent. -- Descartes

What we teach. It is imperative that our nation's colleges design and implement innovative curricula that integrate important topics, along with developing skills in using technology, solving problems, and understanding the sciences and other disciplines. Also needed are interdisciplinary experiences that give students the opportunity to connect their mathematics, especially modeling, to real problems involving aspects of many disciplines. The core curriculum needs to be tied together with student growth threads -- these threads bind together the content among all the required courses as well form the basis for the development of important student attitudes and skills. This core foundation further affords opportunities for undergraduates to progress in their development as life-long learners who are able to formulate questions, research answers, reach logical conclusions, make informed decisions, and study quantitative-based disciplines, such as business, science, engineering, and economics. The undergraduate core program must combine the art, language, science, and problem-solving aspects of modern mathematics. This foundation is critical in the development of the future citizen for the highly technical world of the 21st century. As our sciences and society change, so must our curricula. This connection between science and mathematics will be discussed in this paper.

"The essence of mathematics lies in its freedom." -- Georg Cantor

How we teach. Core-course concepts are constantly interconnected and applied to representative problems from business; professional subjects; computing; physical, social, behavioral, earth, and life sciences; and engineering. Interdisciplinary application problems solved using teamwork are provided to develop student experience in the use of technology, problem

solving strategies, mathematical modeling, scientific reasoning, and technical communication skills (written and oral). In active classrooms, students would develop a curious and experimental disposition; perform critical and creative thinking; and effectively communicate their ideas and results. These problem-solving activities must be performed in disciplinary, interdisciplinary, multidisciplinary, individual, and team settings for progress to be made in the myriad of requirements given to our college graduates during their careers.

New Sciences[1]

New sciences are emerging. And they measure themselves not by any of today's scientific yardsticks, but by the needs of tomorrow's technology. -- Keith Devlin [6]

Educational programs must evolve to meet the future needs of society. However, this does not always occur. In general, academic institutions adopt innovations slowly, sometimes retarding the growth of students. [18] A question to consider is how and when new courses should enter the curriculum. This paper discusses undergraduate core mathematics courses and suggests that a new core course, discrete dynamical modeling, is needed to prepare students for success in the future era of the information age where new sciences like complexity theory and information science will be prevalent.

The mathematical subject needed for understanding the basic underpinnings of these new sciences is dynamical systems. This subject has previously been thought of as an elective, post-calculus course for specialists. We propose that it is time for a discrete version of this course to enter the core as a required first-year course for most students. We briefly present a history of the debut of some core undergraduate mathematics courses, give information about the new sciences (complexity theory and information science), and describe a freshman-level discrete dynamical modeling course.

History of new courses

As is well known, physics became a science only after the invention of differential calculus. -- Bernhard Riemann (1882).

If we look back to colonial America, we find that college-level mathematics was not taught for application, but for exercising of the mind. At the handful of colleges in existence in the eighteenth century, mathematics was either not required or its application was ignored. The one possible exception was geometry, which some professors realized was needed in surveying and, therefore, was taught as a prerequisite course for that profession. At about the time of the American Revolutionary War, civil engineering (a new science) and its applications of designing bridges, canals, roadways, and fortifications needed more than basic geometry. As a result, a new course, descriptive geometry, entered the curriculum. Since not everyone was going to be a civil engineer, descriptive geometry was not required for all students.

Calculus: The first school in America where all students were expected to study engineering was the United States Military Academy (USMA). Shortly after its founding in 1802, students took

[1] This portion of the paper is adapted from an article that appeared in PRIMUS: Arney, D. C., "Educating for the Future: Mathematics for Understanding the New Sciences, PRIMUS, Vol VIII, Sept 1998, pp. 240-252.

algebra, geometry, trigonometry, and descriptive geometry with the intent to use their mathematical skills in their professions. As engineering became more sophisticated and machinery was being designed by mechanical engineers using concepts from physics, calculus was required for understanding this new science. After visiting the Ecole Polytechnique in Paris, the Superintendent of the Academy, Sylvanus Thayer, introduced calculus at USMA. Calculus was required of all students at West Point by 1820 [14]. This new required core course (calculus) was added to the curriculum of many emerging engineering- or science-based schools in the United States. Since then, the number of disciplines requiring calculus has increased. Now, many programs besides engineering require calculus, and it is a core course in many college programs.

College Algebra: While new mathematics was available for inclusion in core programs during the rest of the nineteenth century, only a few new topics entered into college core programs. One could argue that no new science was developed sufficiently to warrant such an action. Therefore, many schools required algebra as the core mathematics requirement and have maintained that requirement ever since.

Probability and Statistics: Operations research, which blossomed during World War II, was the next new science affecting the mathematics curriculum. To use and understand many tools in operations research, full courses in probability and statistics (with topics like least squares) and expanded matrix algebra topics were introduced into the core program. At USMA, a full semester probability and statistics course was required of all students in 1941.

Discrete Mathematics: We didn't have to wait long for the next new science. The war years also produced the computer, and computer science was born. The requisite mathematical topics (propositional and Boolean logic, algorithms, combinatorics, sets, trees, graphs, induction, and networks) were assembled into one course [8,15]. The new science (computer science) produced another new core mathematics course (discrete mathematics).

What will be next? Some will argue that the new science of chaos spawned the need for a new core course in dynamical systems. The development of chaos theory and its associated study of nonlinear dynamics and discrete modeling in the 1970s and 1980s gave rise to a freshmen-level core course called discrete dynamical modeling in the USMA core program in 1990. [16] USMA caught simultaneously the tail-end of the discrete mathematics wave and the leading-edge of the dynamical system wave.

This historical development does not mean to imply that all or even most college core mathematics programs added all these new courses or should blindly follow the model of new science creating new courses. Moreover, we have not mentioned when and how courses leave the curriculum or how they evolve over time. Currently courses containing geometry, some discrete mathematics, some probability and statistics, and some calculus are required in many high school programs. In addition, there has been considerable discussion over the past decade about the content of the mathematics curriculum, especially the make-up of the undergraduate core courses. [7,20,21]

Looking at this history, there seems to be a connection between the development of new science and new mathematics courses entering the curriculum at the core level. Given the important and fundamental role of mathematics in science, this connection makes sense.

Information Science and Complexity

[I] direct my thoughts in an orderly way; beginning with the simplest objects, those most apt to be known, and ascending little by little, in steps as it were, to the knowledge of the most complex; and establishing an order in thought even when the objects had no natural priority one to another. -- Rene Descartes, Discours de la Methode (1637)

It has been mentioned that the new science of chaos is having an impact upon the core mathematics program. Several schools now include dynamic systems courses in their core programs [4, 7, 17]. Let's look a bit deeper into the current situation of new science as we enter the information age.

Through computer processing, we now have machines with access to abundant amounts of data. We easily acquire more and more data. However, we have the problem of transforming this data into information. We do not have the ability to represent it, analyze it, use it, process it, understand it, model it, or improve society with it. Having data locked away in a machine does not mean you have usable information. This is why we need a new science, and this collective enterprise currently is called information science. In many ways, information science is about having machines transform data into information (or knowledge) as the human mind does. We need our machines to perform rudimentary thinking processes. Therefore, to develop information science, we need to understand the complexity of both the human mind and the physical machines holding the data. Then we will be able to use data to our benefit. One of the basic means of modeling the transformation processes of data analysis is through dynamical systems.

At the same time, it has been discovered that many processes and systems in diverse topics such as weather, ecology, biology, engineering, economics, medicine, politics, and warfare, while seemingly unpredictable, do follow deterministic models. These processes demonstrate behaviors and structures that vary from the linear to nonlinear, predictable to random, and discrete to continuous. In reality, very few of the useful systems are simple, linear, or completely deterministic or random. Many phenomena are complex in their behavior: sensitive to slight changes in conditions; quite erratic or aperiodic; and bounded. Such behavior is called chaos. From the abstraction of chaos to other areas of science, many new terms and concepts have arisen: fractals, nonlinear dynamics, cellular automata, computability, artificial life, intelligent systems, knowledge engineering, natural language, learning systems, neural networks, and complexity. In its broadest sense, complexity theory is the collective enterprise of understanding the structure, measure, and classification of these phenomena [19]. Chaos could be considered one part of this new science. Complexity theory discerns that physical, biological, cognitive, and, in general, information systems are complex and need a new science to understand their complexity. Once again, dynamical systems provide a way to model these phenomena. In 1991 a student appreciated this role of discrete dynamical modeling, writing "During this course I have gained some insight into the increasing importance of mathematics in today's world."

Some scientists believe that we are close to a breakthrough in the understanding and utility of complexity theory [10]. Current definitions use a computational approach that defines the complexity of a system as the length of the algorithm for computing the system's model. However, this kind of definition is too ambiguous for precise mathematical measurement and this ambiguity has limited progress in this field. If a better definition can be found, then it might be

possible to compute the complexity of a system. With a suitable new definition we may be able to advance complexity theory to a more productive and persuasive science.

What are the connections between information science and complexity theory? In order to develop information science, we need to understand physical and cognitive complexity to process data into information. Information systems are complex, and new theory is needed to understand and control these systems. This closely related pair of sciences, will provide society its greatest opportunities and challenges as we enter the information age. We need to prepare our students and our society to deal with these circumstances. For this reason, we propose that the mathematical foundation for these two sciences, dynamical systems, be taught in our core mathematics programs. Not only do our future complexity theorists and information scientists need this background, but also our managers, leaders, technologists, decision makers, and informed citizens need it as well. One thing seems certain, complexity theory, information science, and their technical applications will greatly influence our world during the 21st century. [9, 11, 12, 13] Our core educational programs must reflect this requirement.

Discrete Dynamical Modeling

The mind is not a vessel to be filled, but a flame to be kindled. -- Plutarch

The basic concept in discrete dynamical modeling is that the future is predicted by understanding the present and adding to it the hypothesized change over the interval of interest. Discrete dynamical models (often called difference equations) are always solvable numerically by iteration. The prerequisite mathematics to learn and perform elementary discrete dynamical modeling is algebra. Therefore, this topic is accessible for freshmen without an investment in learning more sophisticated calculus concepts needed to study continuous dynamics (differential equations). Many topics, especially the modeling, reasoning, and computing, that are traditionally covered in higher-level courses (junior or senior year) are accessible to freshmen taking an introductory discrete dynamical modeling course. Through discrete dynamical modeling the foundations of our new sciences are available to all students at the core level.

Goals: A valuable set of goals for a core mathematics course might include statements about: students acquiring important and fundamental knowledge for future application; students developing sound, logical thought processes relevant to future science; and students learning how to transfer data into information. By achieving these goals, successful students could formulate intelligent questions, reason and research solutions using new scientific principles, and be confident and independent in their future work. A discrete dynamical modeling course, which includes the study of linear and nonlinear difference equations; systems of equations along with the matrix algebra concepts of eigenvalues and eigenvectors; analytic, numeric, and graphic solution methods and analysis; conjecturing; long term behavior through determination of equilibria and stability; proportionality modeling; and applied problem solving, can accomplish these goals while establishing the mathematical foundation of complexity theory and information science. It is the nonlinear models that can exhibit the property of chaos described previously. Throughout such a course, major mathematical themes can be studied. These themes include using functions (especially sequences); investigating the limit process; examining change; examining accumulation; using vectors; performing approximation; visualizing concepts using graphs; developing and analyzing models; and performing solution methods.

Student Growth: Because student growth and maturity is important in the context of a core course, attention should be given to the course's contribution to development of student abilities and attitudes in areas such as reasoning, problem solving, computing, and communicating [2]. In addition to being the foundation course for these new sciences, discrete dynamical modeling is ideally suited to develop these important abilities and attitudes in our students. Some of the natural opportunities for development of these attributes through this course are as follows:

- Reasoning: Conjecturing solutions and verifying their accuracy are natural processes in a first course in dynamical systems. Often multiple representations (analytic, graphic, numeric, verbal) of structures are possible. The generalization of concepts from specific examples is possible at various places in this course. Additional student motivation occurs from the beautiful fractal geometries that can arise from chaotic dynamical systems. Many students are fascinated by this beauty and strive to learn about its relation to mathematics and science.

- Problem solving: Discrete dynamical modeling is ideally suited for introducing the fundamental concepts of scientific problem solving [2]. Performing the mathematical modeling process of making assumptions, building models, solving models, and verifying the conclusions, contributes to understanding mathematics and its application. Modeling is used to predict or explain changing behavior, such as proportionality or linear growth or decay. Further refinements produce nonlinear equations or systems of equations, which are solvable through iteration. Students can solve applied problems using their skills in modeling, computing, and reasoning. These applications provide motivation for students by showing them the relevance of mathematics in their future lives. A student shared this perspective when writing, "The value of this course is in providing a new way of looking at things that happen in nature and the models used to imitate them."

- Computing: Discrete dynamical modeling can be tied together with a strong computation thread. Understanding the roles, capabilities, and limitations of technology are critical to student success. In this course, students can use computer software (especially computer algebra systems) or calculators as tools for iterating, computing, exploring, visualizing, graphing, solving, simplifying, and integrating various means of problem solving. Recent developments in the calculators and computing software make analysis of systems of equations easily accessible to undergraduate students. Computers and calculators are natural tools to help students in most stages of dynamical modeling.

- Communicating: A dynamical modeling course provides many opportunities for students to grow in their communications skills--expressing ideas clearly and effectively using proper mathematical notation.

How do you do this? Since 1990, all students at West Point have taken a discrete dynamical modeling course containing the material described above [1]. While students at West Point are often highly motivated, they are not exceptional in their preparation, aptitude, or interests. They, like all undergraduates, need to transition from high school thinking to the higher level of college learning. This introductory discrete dynamics course is perfect for performing that transition. At West Point, it is the first course in a four-course sequence containing additional integrated courses in calculus, differential equations, and probability and statistics. In order to provide this

opportunity, we had to delay and reduce the content of the calculus courses. We still teach enough calculus in the first year to get our students ready for the study of physics during their second year. We use more computation and cover fewer analytic techniques to compress our calculus program. We have found that the introduction of calculus topics is more efficient because our students have studied the discrete analogs of many of these concepts during their study of discrete dynamical modeling. In 1994, a student in calculus remarked "The calculus seems a lot less difficult because we have already seen difference equations and how to apply them to the world we live in. I personally believe that mathematics seems a lot less harmful when it can be put to use."

Over the years, the Academy's discrete dynamical modeling course has evolved. Compared to 1990, we now cover more modeling (but fewer topics), more conjecturing (with less memorization), and more numerical solving--iteration (with fewer analytic solution techniques). Homework and quizzes often call for students to conjecture forms of solutions or discover phenomena like resonance or limit cycles. Application problems and larger lively interdisciplinary projects require students to model and analyze the dynamic behavior of monetary accounts, voting trends, market shares, populations, biological systems with predators and prey, chemical reactions, and heat flow. This first-year, core discrete dynamical modeling course is accessible, valuable, and needed for all students, even those who will not continue their study of mathematics. [3,15,16] We believe that our most important course for the future development of our students is the discrete dynamical modeling course. This is the course that we would teach, if we had only one required course in our core program.

Conclusion

All the pictures that science now draws of nature ... are mathematical pictures. -- Sir James Hopwood Jeans (1930).

Since our educational programs and courses should be designed for the future needs of students, we propose that a course in discrete dynamical modeling enter the core as a required mathematics course for most undergraduate students. This course would prepare students for success in the information age through the understanding of the basic underpinnings of the new sciences of complexity theory and information science. It provides opportunities for students to mature and grow in solving problems and modeling behavior. Discrete dynamical modeling is accessible to first year students, who need for their future success the reasoning, modeling, computation, and language of the new sciences of the 21st century and the modeling and problem-solving skills to lead society.

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