First Two Years of Mathematics for Scientists and Engineers

Jeffrey E. Froyd, Visiting Professor of Electrical Engineering
Zachry Engineering Center, MS 3128
Texas A&M University,
College Station, Texas 77843-3128
Jeffrey.E.Froyd@Rose-Hulman.edu

Consider the following proposition “In the long run, your only sustainable source of competitive advantage is your ability to learn and create faster than others.” The proposition is the individual analog to the proposition that Peter Senge advocates for organizations, “In the long run, the only sustainable source of competitive advantage is your organization’s ability to learn faster than its competition.” The rapidly growing amount of information also supports the proposition. People have speculated that the half-life of the knowledge that a scientific or engineering graduate gains during a bachelor's degree is 2-5 years. Then, after ten years at least 75% of the factual database that the graduate gained in school is obsolete and must be augmented by new knowledge gains in the workplace. Although the correctness of the proposition could be debated at length, let's accept the proposition as true for present purposes.

If true, then it is not what the graduate knows when he/she graduates that is most important, but the rate at which the graduate can learn and create after graduation. Therefore, in discussions about content and whether to include topic X, the criterion for inclusion should not be "Any graduate must know topic X." or "Topic X is fundamental to the study of mathematics, physics, chemistry, engineering, etc." Instead, the criterion that follows from the proposition is the following, "To what degree does topic X increase the capacity of a graduate to learn and create?" With temporal constraints imposed by a four-year degree, the criterion could be restated, "What limited set of topics provides the largest increase in the capacity of a graduate to learn and create in his/her chosen field of study?" The difference between the two types of criteria is crucial because the list of topics that satisfy the criterion "Topic X is fundamental to the study of mathematics, physics, chemistry, engineering, etc." is very long and growing. Using the "Topic X is fundamental" criterion encourages long, highly charged debates because every faculty member has her/his set of topics that are fundamental and they are all correct. Since the union of the sets of topics from three or four diverse faculty members probably exceeds the limits of a four-year curriculum, debates on topic choice using "Topic X is fundamental" will never conclude. A different criterion is needed and a criterion based on the capacity of a graduate to learn and create is proposed.

What are mathematical topics that increase the capacity of graduates to learn and create? It is suggested that the topics that most increase the capacity of graduates to learn and create are related to the abilities to

- envision desired behavior and physical implementations that realize the desired behavior,
- abstract behavior that they observe in the world around them,
- formulate these abstractions carefully and quantitatively,
- predict behavior from the quantitative models, and
- apply the knowledge of the predicted behavior to improve their understanding of and their ability to predict behavior in the world around them.

Topics related to the ability to start with one symbolic formulation and derive symbolic formulation do not seem to increase the capacity of graduates as much as topics related to the

previously listed abilities. Are such topics such as the ability to derive symbolic indefinite integrals important? Absolutely, but importance is not the agreed upon criterion for inclusion. Instead, the criterion is the ability to increase capacity to learn and create.

Let's now explore the consequences of using the criterion the build the content of a mathematics curriculum. Does the knowledge of the concept of a function increase the capacity to learn and create? It is suggested that the answer is only slightly. Knowing the definition of a function may allow students to pass a test. However, knowing the definition does not appear to increase the capacity to learn and create as much as the ability to abstract observed behavior or measured data by constructing functions to capture the essence of the behavior or the data. For example, can students construct functions that model a quantity to be optimized? Can students create functions that describe motion or other observed behavior? Does knowledge of symbolically constructing derivatives of expressions increase the capacity to learn and create? Again, it is suggested that the answer is only slightly. This is especially true since we now have software that performs symbolic manipulation faster and more accurately than humans. Instead, students that can apply the derivative to create quantitative abstractions or models, e.g., differential equations, have the greater capacity to learn and create. Similar observations can be made about computing limits of expressions, sequences, and series; symbolically computing integrals; or solving differential equations. What graduates need to increase their capacity to learn and create is the ability to express questions mathematically, reason with the created abstractions, and translate mathematical results into powerful ideas. In other words, students must use mathematics as a second language (MSL) to describe, understand, and predict behavior in the world around them.

Similar content questions can be explored other areas of mathematics. Is it more important to express the idea of getting close to something or to express relationships between rates of change in a system? Is it more important to express the idea of finding closed-form expressions for integrals of symbolic expressions or to express the idea of adding acceleration to get velocity, velocity to displacement, force dotted with displacement to get work, electric field dotted with displacement to get potential, [magnetic field], etc. Is it more important to express the idea of determining whether and where a sequence of numbers ends or to express the idea of determining quantitative models for a set of data? Is it more important to determine area under curve or create a sum that estimates the weight of a column of air in the earth's atmosphere, or a sum that estimates work done along a curve, or a sum that estimates gravitational force due to a region or matter. In the spirit of an interdisciplinary approach, it is suggested that communities of mathematicians, scientists, and engineers should formulate answers to these and other questions. Once, a list of important ideas to express mathematically have been developed, then the next task is to order the list in which students will learn to express these ideas and finally to develop activities through which students can gain expressive fluency.

Let's consider another example in which we can apply our criterion in selecting and organizing the content of a mathematics curriculum. After graduation direction, people frequently encounter sets of data. Data may be offered to support a proposition or data may be measured behavior of a system. People then must either interpret the data to determine the degree of support that it offers for a proposition or use the data to construct a model for the system. It is suggested that the capacity to learn and create will be significantly increased by the ability to interpret sets of data or use them to build models. Methodologies that support interpreting data or empirical modeling fall within probability and statistics. Therefore, calls are made to include a course in probability and/or statistics. However, faculty who teach these courses point that students who enter the course have little or no previous exposure to concepts of randomness and risk. Further, they describe the difficulty in changing the deterministic orientation of students in a single course. A traditional answer is multiple courses, but this answer encounters many obstacles in already overloaded curricula. An alternative approach would be to introduce randomness and analysis of data early in

a mathematics curriculum and spirally build concepts over the course of the curriculum. The integrated approach appears to answer the difficulty in helping students construct a stochastic framework in a short period of time and providing them increased capacity to learn and create.

Once we have agreed that the criterion for including a topic in our two-year mathematics curriculum is increasing capacity for a graduate to learn and create, is there a compelling reason to limit the possible topics for inclusion to calculus and differential equations? Isn't it possible that topics in linear algebra, discrete mathematics or statistics will give graduates more capacity to learn and create than simply including more topics from calculus and differential equations? Physicists, who have worked on a related approach to the introductory, one-year physics curriculum, have developed a curriculum called "Six Ideas That Shaped Physics." These ideas are

- Conservation Laws Constrain Interactions
- The Laws of Physics are Universal
- The Laws of Physics are Frame-Independent
- Electromagnetic Fields are Dynamic
- Matter Behaves Like Waves
- Some Processes are Irreversible

Choice of these topics met the guiding principles of the committee that developed this approach: a) less is more, b) include twentieth-century physics, c) use a story line, d) pay attention to the results of educational research, and e) seek the middle way. Although every topic from previous physics curricula may not fit under these six ideas, physicists that developed this curriculum thought that fluency with these six ideas would increase the capacity of graduates to learn and create in the future. Following the lead of these physicists, what are the six ideas that shaped mathematics?

Gary Sherman, Department of Mathematics, Rose-Hulman Institute of Technology, describes four powerful ideas that shape his view of mathematics: 1) measurement, 2 measurement in the presence of structure, 3) equivalence, and 4) transformation (go someplace, do something, return). He presents an informal explanation of the four ideas as they relate to discrete mathematics.

> Counting (measurement) in a raw set (of real-world size) is a mind-numbing, indeed computer-numbing, chore because you must exhibit the subset in question and then count it. "Natural", or created structure (graphical, combinatorial, algebraic, or some combination of the three), enables one to count the set without iterating the set. For example, how many permutations are there of three symbols? Easy, list them 1,2,3; 1,3,2; 2,1,3; 2,3,1; 3,1,2: 3,2,1 and count them. How many permutations are there of 52 symbols? Well, the functional structure of permutations enables one to create the structured count 52! without listing anything. A more complicated example is to measure how long it takes to shuffle a deck of cards. Indeed, you can't define precisely what you mean by the question until you impose group structure on the set of permutations that model the shuffling process. And that is a reason, among others, that I start discrete mathematics with a two-day class discussion about what it means to shuffle a deck of cards.
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> Once you start counting in structure you almost always discover that your answer depends on some hidden or confusing or implicit equivalence relation. How many permutations are there of 3 symbols? Six you say? Not hardly. Any "sophisticated" mathematician will tell you there are only three: id, (a,b), (a,b,c); i.e., the three are: the identity, a transposition and a three-cycle, because there isn't a dimes worth of structural difference between the transpositions (1,2), (1,3), and (2,3) nor is there a dimes worth of

difference between the 3-cycles (1,2,3) and (1,3,2). If you care only about parity, there are only two permutations (of any given degree) --- an even one (id, or (1,2,3) or (1,3,2)) and an odd one ((1,2) or (1,3) or (2,3)). And on and on. What's really going on in any equivalence relation is that the equivalence classes are just the orbits of some group of permutations acting on the universe in question. (Sometimes this is of practical use and sometimes it isn't, but it is always in the background.) For example, (1,2) and (1,3) are naturally equivalent because they are conjugate. That is, the symmetric group on three symbols (S_3) acts on itself by conjugation ($x^g = gxg^{-1}$) and under this action, really a group of permutations of S_3, (1,2) and (1,3) are in the same orbit because (1,2) = (2,3)(1,3)(2,3)^-1. A way to look at this: to invoke the transposition (1,2) ---- don't! Rather go someplace (= (2,3)), do something else (=(1,3)), come back (=(2,3)^-1). Indeed the mantra all students should be reminded of incessantly is "go someplace, do something else, come back" --- the essence of transformation and in some sense, for me at least, the most important idea of theoretical and applied mathematics. Never solve a hard problem find a point of view which makes the problems solution conceptually obvious: don't multiply, add; don't use that messy matrix, diagonalize it; don't convolve, use an inner product and on and on.

And of course what's really behind my "go someplace, do something else, come back" mantra (modulo some algebraic technicalities) is the notion of commutativity (and we are back to algebraic structure); i.e., is AB = BA, where this equation is to be taken with a grain of salt (here is where I'm moding out the algebraic technicalities associated with unary functions and binary functions for the sake of pedagogical impact). Wouldn't it be wonderful if we lived in the commutative world all beginning undergraduates do? They are trained to mindlessly invoke "computational tricks" like a+b = b+a and ab=ba until they come to believe that

flarn clarp = clarp flarn

is a truism to invoke, rather than a question to ask: Is the flarn of a clarp the clarp of the flarns?

Is a root of a sum the sum of the roots? Is the power of a sum the sum of the powers (in particular, is the reciprocal of a sum the sum of the reciprocals?)? Is the integral of a sum the sum of the integrals? Is the derivative of a product the product of the derivatives? Is the log of a product the product of the logs? When the answer is yes, we are in high mathematical clover. When the answer is no, sometimes we are stuck --- but sometimes we are actually in higher mathematical clover because we can find a tworble so that

flarn clarp = tworble flarn;

i.e., clarp = flarn^-1 tworble flarn or flarn clarp flarn^-1= tworble

and the conjugation is less expensive that the original computation. So, the mantra "go someplace, do something else, come back". The students have a history with this idea (although they don't realize it without some serious provocation):

log product ≠ product log

but

log product = sum log

so

product = log ^-1 sum log.

(A notational comment: to make the mantra work you must read the right hand side from right-to-left because of the analyst's penchant for writing functions to the left of their argument.)

Current two-year curricula in mathematics emphasize calculus, differential equations and, perhaps, statistics. As a result, students often complete two-year of intensive, high-quality mathematics with, at best, minimal knowledge of mathematics that has been developed in the past hundred years. Where are the powerful mathematical ideas that have been developed in the twentieth century? In number theory? In algebra? In quantum mechanics? Which of these ideas will increase the capacity of students to learn and create? How are these ideas related to calculus, differential equations, and statistics? How can powerful new ideas in mathematics be merged with existing ideas in calculus, differential equations, and statistics to create a two-year curriculum that will increase the fluency of students in mathematics and increase their capacity to learn and create? Once a set of powerful ideas has been identified, next we need to describe a set of activities, which students sufficiently fluent with these ideas, will be able to master. Now that we have identified the outcomes for a two-year curriculum in mathematics, let's consider ordering topics.

Currently curricula in calculus, differential equations, and statistics emphasize a topical order that develops ideas in an order so that students do not have to use concepts that they have not explored thoroughly (although many will disagree with the adverb). So, concepts of limit and continuity must be introduced and explored before the derivative can be examined. Ordering mathematical topics via topical precedence is logical, but students may struggle cognitively and motivationally with where to attach concepts such as limits and continuity because they cannot see the big picture. Efforts in reform calculus have offered substantial changes in the order, but the emphasis remains on thorough understanding of prerequisite topics before introducing the next topic. However, after graduation, scientists, engineers and mathematicians often learn only fragments of an area in order to work on their current problem. In order to increase capacity to learn and create, it may be more desirable to order ideas around questions that could be attacked. The processes in which students participate can be as important or more important than the ideas that are presented to students. Let's consider curricula in which order is generated from the questions that students will be asked to engage.

People learn what they have opportunity to learn. If students work in a classroom environment where the teacher presents information about powerful ideas, offers examples, and provides straightforward homework and examination activities based on the examples, then students will learn to do straightforward activities. In general, they learn mathematics, science, and engineering: posing problems, thinking about different ways to approach a problem, discovering connections, because they aren't doing mathematics, science, and engineering. Offering challenging homework and examination activities is an inadequate substitute, because students may have backgrounds in which they have not tackled a sufficient number of these types of challenging activities. Therefore, they are unsure about how to proceed, can become easily frustrated, and fail to take advantage of the opportunity to learn. Once we have our powerful ideas, let's consider carefully questions that can be posed in class to engage students in grappling with these powerful ideas. However, these questions must be developed carefully. "The student must find the question natural and interesting. If you find the question natural and interesting, they might and if you don't, they won't."[1] Ultimately, what students do is what they will learn. If they participate in the process of tackling interesting questions and applying powerful mathematical ideas, then they will increase their capacity to learn and create.

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[1] Gary Sherman, Department of Mathematics, Rose-Hulman Institute of Technology, private communication

What are natural and interesting questions? Given the price of stamps since 1900, what is your prediction for the price of stamps in 2015 and how do you support your conclusion? Given two interacting species how do populations of the populations of the species vary over time? What is a model for the data? What patterns are present in a set of data? How fast is the car traveling? How do you build a guidance system? What is a magnetic field? How much energy does it take to add two one-bit numbers? What is the age of the universe? What does a filter do to sound? How does a computer work? I'm sure you have devise better questions than my short list.

What do we include? Within a limited time, we include a set of topics that maximizes the increase in the capacity of graduates to learn and create. What are the topics that provide the largest increase in the capacity to learn and create? Our experience suggests that the answer is those topics that promote the ability of students to quantitatively describe, understand, and predict behavior in the world around them. It is also suggested that what is required is a synthesis of ideas that include twentieth-century mathematics. Don't be limited to the currently accepted silos of calculus, physics, differential equations, algebra, number theory, chemistry, statistics, biology, linear algebra, etc. Are these silos important? Yes, but science and engineering students don't have time to take traditional courses in all these areas. Do we just sigh and say, "But that is how the material has to be arranged, because that is how the material is currently arranged." Science and mathematics students will have to make do with small portions of what could be offered. Or do we think hard about the wealth of mathematical ideas and their relationships to behavior in the world around us and synthesize a set of powerful ideas that fit in the allocated time? How do you present these topics? We synthesize sequences of natural and interesting questions to engage students in doing mathematics. To the questions posed about the goals and content of mathematics, three answers have been offered: 1) focus on increasing the capacity to learn and create, 2) synthesize ideas to build new, powerful ideas that increase the capacity to learn and create, and 3) develop sequences of natural and interesting questions to engage students.