Goals and Content of a Two-Year Mathematics Curriculum

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COMAP

During the first two years: *What shall we teach and how shall we teach it?* Two years is a significant amount of time in the development of a student. At the end of the sophomore year, will students have progressed sufficiently to read and understand scientific exposition independently? Will they be able to research and learn mathematics on their own?

Overview

The first two years of college represent a crucial stage in the growth of a student. During this period the student is expected to transition from the high school classroom to the environment of upper divisional courses in applied science and engineering, where typically they are expected to learn and apply relatively sophisticated mathematical exposition as an implied part of learning science and engineering. Further, they are expected to assimilate the information in an increasingly independent fashion. For most science, mathematics, and engineering students, the only department teaching in each of the first four semesters is mathematics. Obviously, the content of these four courses is important if the courses are to provide the necessary mathematical foundation. But perhaps more important is the role these courses play in the development of a life-long learner. How does an educator provide for the progressive development of a student during the first two years?

Student Growth Dimensions

Among the areas needing careful attention during the first two years are:

Learning how to learn Mathematics is the language of science. My experience is that few freshmen come prepared to read mathematics independently. Yet, beginning sophomore year or earlier, other disciplines will rightfully assume a proficiency in reading and writing mathematics as an outcome of the student's previous mathematical preparation.

Communication Key to mastering the language of mathematics is expressing oneself comfortably in the language, not merely regurgitating definitions, theorems, and formulae that have often been memorized with little real understanding.

Mathematical Sophistication Critical to a student's progressive development in mathematics is a curriculum organized so that it progresses in degree of sophistication of the underlying ideas. This is not simply a question of prerequisites – some ideas are harder for students to master. If students are to learn in an increasingly independent manner, then attention must be paid to the ordering of the topics. For example, the simpler ideas of matrix algebra may be introduced early in the curriculum, while the more sophisticated ideas of linear algebra are delayed. Similarly, if students are not comfortable with the notion of a finite difference, it is not likely they will understand the concept of a limit of a difference ratio.

Modeling Students gain greater confidence and appreciation of applied mathematics if they can use the mathematics they are learning to approximate the world about them. Gaining an understanding of how the modeling process facilitates the use of mathematics to represent rather complex behavior is excellent preparation to read and understand scientific exposition. Further, modeling can be used to motivate the learning of new mathematics as the curriculum unfolds, connecting what otherwise may appear to the student to be disparate mathematics. For example as the curriculum unfolds, simple linear, deterministic, discrete models can be refined to consider nonlinear, continuous, and probabilistic considerations as appropriate.

Technology The first two years provide an opportunity to prepare students to solve interesting applications using technology. If designed properly, this experience can shed greater understanding on the underlying mathematical concepts while enabling students to solve problems not tractable without technology. Mathematics can be more exciting if technology removes some of the tedium often associated with courses such as calculus. Finally, used creatively, the visualization of key mathematical concepts can be enhanced remarkably with technology.

Connectivity The mathematics presented during the first two years should reveal connections to the larger curriculum. Most students want to know how the disciplines are connected, especially in courses they are concurrently taking.

The History of Mathematics Many students find that learning the *who, why, when*, and *how* as well as the *what* in the creation of mathematics is interesting, and often motivational. For some students, it makes mathematics come alive. Students should understand that mathematics is an ongoing human endeavor. Reading and research projects in the history of mathematics provide further opportunities for students to practice communication skills.

Content Objectives

The needs of all disciplines in the overall curriculum are important considerations in the development of the two-year mathematics curriculum. But providing the opportunity to present the selected concepts in a coherent and mathematically sound way is crucial to the ultimate design of an experience that permits progressive student growth in each of the growth dimensions enumerated above. Given the opportunity to examine the tradeoffs inherent in designing the two-year experience, I believe each institution would design distinct curricula. For the purpose of discussing how to provide growth opportunities, the following content objectives are offered. The choice is based on a desire to present a *breadth* of applicable mathematics: discrete and continuous; deterministic and stochastic; and linear and nonlinear. The objectives selected for each of the following topics are based upon a hypothetical 15-hour curriculum, and my understanding of numerous "give and take" discussions among mathematicians, scientists, and engineers over the years. Such discussions are necessary for a core mathematics experience that is both useful as preparation for other disciplines and mathematically sound. Participation by other disciplines in the design of the core mathematics experience increases the probability that key mathematical concepts will be reinforced throughout the curriculum while fostering a sense of ownership and cooperation across the faculty.

Discrete Mathematics: Difference equations through system of difference equations. An ideal subject for integrating modeling, technology, and contemporary mathematics building upon high

school mathematics. Furthermore, an understanding of finite differences and finite sums is an excellent preparation for understanding calculus.

Matrix Algebra: Ideally, matrix algebra through the eigenvalue-eigenvector problem can be integrated early to be used immediately to solve systems of difference (or differential) equations. The higher concepts of linear algebra can be revisited later in the curriculum.

Calculus: Through multivariable calculus for all students. The end goals for coverage of vector integral calculus may vary among the particular disciplines being serviced by the core curriculum. Depending on when majors are declared, some tailoring of the multivariable curriculum may be possible.

Differential Equations: First and second order constant coefficients with the types of forcing functions normally seen at the undergraduate level. If matrix algebra and systems of difference equations are presented, a study of systems of differential equations extends the student's power to model real behavior while reinforcing important fundamental mathematical concepts.

Probability and Statistics An introductory experience in probability and statistics is essential to provide a broad overview of the mathematics we use to model our world. Minimally, the concepts of the enumeration of sample spaces, conditional probability, independent events, expected values, and decision making under uncertainty should be included. A brief introduction to descriptive statistics and curve fitting should also be part of the introductory curriculum. Finally, a brief introduction to Monte-Carlo simulation enhances immensely the student's ability to incorporate the element of chance in their analysis, while aiding the understanding of probability.

Model Curricula

Examples of how several schools have selected combinations of each of the above content areas while integrating student growth objectives appropriate to their institutions occur at West Point, Carroll College, and Harvey Mudd College. Each two-year curriculum differs in the final objectives for each of the content areas, and the amount of "weaving" or "revisiting" of the content areas.

Providing Growth Opportunities

Student growth in each of the growth dimensions can be planned for whatever mathematical content is finally selected for the first two years. For each dimension, it is useful to select reasonable "end outcomes" and then select achievable and measurable intermediate objectives that occur as the curriculum unfolds. Viewed in this manner, the objectives for each dimension are planned "down" the curriculum. The curriculum can be viewed as a matrix, or fabric, with the growth objectives woven with the content dimensions. For example, in the modeling dimension, one may select final objectives that include building models of behavior in discrete and continuous environments. Eventually, some of the models may need to be refined to consider chance, or perhaps a more accurate model that employs nonlinear mathematics requiring numerical approximation. If the first course treats difference equations, the concept of proportionality can be reviewed as elementary constant coefficient difference equations are built. At that point problems such as pollution with dumping occurring at discrete intervals can be modeled. The models can be refined to include connected multiple pollution sites when systems of difference equations are

studied. As the curriculum progresses to calculus, the question of continuous dumping can be treated using differential equations and systems of differential equations. Students see that the mathematics they are learning extends their ability to model the world about us. Such an approach provides additional threads to connect the curriculum internally (the mathematics curriculum) and externally (the other disciplines). The external connection can be further enhanced with input, and perhaps participation, in the design, presentation, and critique of the modeling projects. Finally, the projects provide opportunities in communication, technology and the history of mathematics. A similar approach would be followed for each of the other threads.

The Role of the Instructor

As the above curriculum unfolds, the role of the instructor changes. Perhaps as a transition from the high school environment, the instructor initially is like a "coach", providing the initial mathematical "training" while urging students to greater achievement. Quickly, however, the need for counseling becomes greater as the instructor must continually assess every student's status in each of the growth dimensions and design opportunities for students to take their next step. Later in the curriculum the instructor becomes more of a "travel guide" than coach, pointing out the interesting places to visit as the students begin to learn more independently. If we ask the students to learn in an increasingly independent manner, than we must provide the motivation to do so. What motivates a particular student and how do you "hook" them – is it history, modeling, mathematical rigor, using technology, or connections to other disciplines? While perhaps not as glamorous as giving an eloquent lecture or being the source of knowledge in the classroom, guiding student growth may be more be beneficial to most students in the long run.