A Continuum of Choice: Instructional Techniques in Undergraduate Mathematics

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In 1995, when doing an initial assessment of the first decade of calculus reform, Leitzel reported that although some content in calculus courses had changed, the more apparent and widespread change was pedagogical [1]. Instructors were making changes in undergraduate mathematics' classrooms in an effort to increase the emphasis on conceptual understanding and decrease emphasis on symbol manipulation. Many schools were making use of graphing calculators and incorporating modeling and applications into their courses. Some schools began to replace the traditional undergraduate large-section mathematics lecture class with smaller sections in which students experienced one or more of a variety of instructional practices. In recent years, pedagogical changes have continued in undergraduate classrooms.

Constructivism has been the major guiding philosophical basis for pedagogical change within mathematics classrooms in the last decade. One can trace the roots of constructivist ideas to general research in cognitive psychology done in the latter half of this century, and more specifically to the work of Piaget. Piaget believed that action and knowledge are inextricably linked [2]. Constructivists generally agree that all knowledge is constructed by the learner, and that cognitive structures are under continual development. Constructivists believe that it is purposive activity that induces the transformation of existing structures, and that the environment presses learners to adapt [3]. Once an instructor acknowledges a constructivist perspective as a cognitive position, "methodological constructivism" follows. "Once a constructivist perspective is adopted, the day-to-day life in the classroom is profoundly and significantly altered for both teacher and students [4, p. 314]." One does not, however, need to understand the deeper cognitive aspects of constructivism to teach in what would be considered a constructivist manner. Many teachers are natural constructivists, guided by what they feel is most beneficial for their students: a more interactive and engaging classroom.

Through the constructivist lens we see an obvious distinction between lecture or traditional teacher-centered methods and active learning or student-centered methods. One can picture a continuum of classroom environments in which a variety of methods reside. On the one extreme there are classrooms in which only the teacher speaks or lectures. On the other extreme, a teacher may not even be present as the students independently and individually work through the material. (I propose neither of these extremes used exclusively serve the students well.) As one moves from the lecture to the independent study, varying degrees of action by, and interaction between, teachers and students occur. Methods which may influence movement along the continuum include the use of questioning or discussion, (applied) problem solving, technology, explorations and discovery, multiple representations of mathematics, writing, various types of assessment instruments, smaller section sizes, and collaborative or cooperative groups. Gradations of student-teacher action and interaction occur within these methods spreading various adaptations of each along the continuum as well. In using a continuum image, instructors can select general strategies or methods and

implement them within their comfort level. Although there may be additional methods, I will consider the nine listed above in addressing some of the issues and future direction of instructional methods in undergraduate mathematics.

What are the strengths/drawbacks of different instructional methods?

Questioning/Discussion. The first step toward a more active and interactive classroom is to let students enter the mathematics discourse. We can accomplish this, for example, by the instructor questioning students (Socratic type interaction), by the instructor turning students' questions back to other students, or by the instructor allowing the students' questions or discussion to control the flow of the class. Although varying in their degree of being student- or teacher-centered, these examples illustrate that in place of the instructor doing all of the speaking, he or she facilitates the students in actually presenting the material. The advantage of this method is that students are forced to articulate ideas in their own words. Other students may understand these explanations better. The student who is speaking is not only required to mentally think about, or perhaps even mentally image, the idea, but also must verbalize it. Student talk is very revealing of students' understanding [5]. Instructors can learn a great deal from what a student says. In this type of classroom students may arrive more prepared if they know the instructor expects some degree of participation.

Some instructors voice concern about opening one's classroom to discussion and questioning as it may take control away from the instructor. Allowing student participation in classroom discourse takes more time and class discussions may head down a path not planned by the instructor. Many teachers consider this a strength as students take more responsibility for their learning. These discussions may lead to less material being "covered" than in a traditional lecture. In this environment, instructors must pay very close attention to what is being said and how it is said. Poor articulation by one student can lead to the development of misconceptions by other students.

Problem Solving. Students are interested in the worth of the mathematics they are studying. The mathematics studied by undergraduates in their first two years provides a bridge between high school algebra and engineering and science. Problem solving activities, and especially real-world applications, provide students a picture of how the mathematics they are learning is used. Toward one end of our continuum is supplemental problem solving. Textbooks usually provide some supplemental problem solving activities in the list of problems at the end of each section. These begin to help students see the uses of the current topic under study. A step further using this method is to integrate applications into the course. Application problems can serve as the vehicles for learning concepts. Real world scenarios with realistic data often motivate students. Real world problems offer some degree of uncertainty from which instructors can discuss the concepts of assumptions and accuracy, as well as the appropriateness of various mathematical tools. As one moves toward more student autonomy, instructors can use interdisciplinary projects to provide a realistic and fully integrated problem solving approach. These type projects require students to model situations from another discipline mathematically, choose from an array of solution techniques and translate their solution to address the original problem. These projects benefit both students and teachers in learning more about the use of mathematics by other disciplines.

The largest difficulty in incorporating problem solving is time. Students must be given time to wrestle with uncertain situations. Although having students involved in the modeling process is very beneficial, when integrating applied and interdisciplinary problem solving into a course, time must be

allocated for this endeavor. Instructors should give compensatory time to students for major project submissions. In making a decision to truly integrate problem solving, one must make the difficult decision of what to delete from the course syllabus. Additionally, developing real-world interdisciplinary projects is no small undertaking for faculty members. Coordination between faculty in partner disciplines is professionally worthwhile, but also time consuming.

Technology. Many calculus reform projects focused on technology use. These projects largely looked at integration of calculators and computer algebra systems (CAS) into calculus courses. Instructors looked at technology to provide the time needed for other activities through quicker calculations and graphing. Instructors have incorporated technology in varying degrees. On the one hand, instructors can simply allow calculator use in the classroom, while on the other they can demand every student own and use a particular calculator or CAS. In some cases instructors are teaching/facilitating courses exclusively through or with a computer (see Exploration and Discovery method). Technology has allowed a refocusing on conceptual understanding versus procedural proficiency. Students can develop a model or set up a problem and then use their hand-held calculator or CAS to solve the model. Hand held technology offers quick and easy generation of numerical data and graphs in the classroom. Student can perform symbolic manipulation using technology rather than a pencil. Calculators now provide many symbolic analytic solutions. Instead of spending time generating data, plotting points, or performing algebraic manipulations, students can spend time analyzing and understanding the model phenomena. Both the current available calculators and CAS provide essential support to students' applied problem solving efforts discussed above. Technology offers students the ability to explore and discover.

Concerns over fundamental skills linger. What should students be able to do without a calculator? What is the importance of hand calculations/skills? Which concepts/skills should be learned first by hand, but once understood shifted to a calculator procedure? When do we introduce CAS? With current calculators, when do students really need a separate CAS on a computer? How do we make available technology a tool and not a crutch? In only a few short years our students will be the middle school children of today who have grown up with technology. Their view and use of technology will be very different than that of even today's undergraduate. The mathematics community must think deeply about these questions as we prepare to educate these students in the twenty-first century.

Explorations and Discovery/Facilitated Laboratory. In some classrooms technology is added on to the course curriculum. In laboratory classrooms technology is integral to the course. Examples of laboratory type courses include Calculus Using *Mathematica*; Calculus, Concepts, Computers and Cooperative Learning (C^4L); Calculus and *Mathematica*; Calculus in Context; and Project CALC: Calculus as a Laboratory Course [6]. Those involved in the development of these courses believe that technology is not a solution to pedagogical problems, but offers alternative choices to address those problems. A comparative study between a traditional course and a Calculus and *Mathematica* (C&M) course at the University of Illinois showed the C&M students obtained a higher level of conceptual understanding while maintaining computational proficiency, and also showed a more positive disposition toward mathematics and computers [7]. In laboratory course students discover mathematics through exploration, conjecture and verification using technology. They also program computers to perform procedures. In doing so, students develop understanding of the processes involved. Researchers in the C&M study concluded students benefited by better visualization of ideas which promoted sound conceptual understanding, discovery

learning by exploration which induced reflection and resulted in developing relationships between concepts, and cooperative learning which established rapport and teamwork among the students [7].

One of the concerns of laboratory courses is that students will blindly use the technology without understanding the concepts (see Technology). This is preventable if the laboratory course is carefully designed and facilitated. Assessment instruments must be designed that measure both conceptual and procedural understanding, not just technological proficiency. Laboratory courses also are time consuming. Discovery and exploration require time for conjecturing, testing, reformulating and reflecting. A final concern for some schools is that computer laboratory facilities necessary to offer a laboratory course are not available. The alternative, to have students purchase laptop computers, may also not be feasible.

Multiple Representations. The "rule of three" was introduced during calculus reform efforts based on the belief that emphasis should be placed on numerical, graphical and analytical representations of mathematics. Since its initial introduction some educators have moved to a "rule of four" to include the representation of mathematics in words (either verbally or in writing). Focusing on various representations assists students in understanding and making connections between concepts. Additionally, these representations provide students with weak algebra/symbol manipulation skills the opportunity to grasp the concepts while they hone their procedural skills. Student participation in the classroom and writing in mathematics support the concept of representing mathematics with words. Technology has opened the floodgates for various representations of mathematics. Students can quickly obtain a graphical or numerical representation of a particular function. They can begin to analyze its behavior immediately. Students can observe and describe long term behavior and examine limiting processes. CAS has provided analytic support for the student and made it easy to examine simultaneously the analytical, numerical and graphical representations of problems and solutions.

There are very few drawbacks to incorporating multiple representations of mathematics in the classroom. One, however, is students' acceptance of one representation as "proof." As core or service course mathematics classrooms have shifted away from formal proofs, some students may believe all representation are equal and will use one example, or in this case one representation, to generalize or draw a conclusion. For example, a student's conclusion from analysis of a particular graphical solution may be incorrect due to the domain in which the graph was viewed. Additionally, many times in undergraduate mathematics we do not have graphical or numerical solutions, but truly have graphical or numerical *approximations* to solutions.

Writing. More than just a fourth representation of mathematics, student writing truly reveals students' understanding. Having students articulate their interpretation and analysis of mathematical concepts and problems is extremely revealing. In turn this provides feedback to the teacher for instructional responses. Writing prepares students to clearly communicate mathematics to clients or outside reviewers of their work. It forces logical and organized thought. One can integrate writing into a course in varying degrees. Exams or quizzes can include short answer or explanation type problems. Given a scenario and a graph, for example, students can describe the link between the behavior of the graph and the scenario. After using an approximation technique, students can discuss whether or not their solution is reasonable. Instructors can assign essays that require students to explain a course concept to perhaps a friend who is not taking the course. The use of

journals has become more common in mathematics courses. Student journals may include concept analyses, reflective summaries, annotated problem solving or essential term definitions.

Developing good writing requirements is difficult, and assessing written work is equally as difficult for most beginners in this area. We, as well as our students, will need time to develop as we incorporate writing into mathematics. Time again becomes an issue to consider. Just as various types of analytic problems require varying time efforts from students, time required for writing varies with the requirement. All writing requires some time for reflecting and organizing one's thoughts before composing the prose. Assessing writing requirements also involves instructor time; we need to carefully listen to (read) what our students are saying.

Assessment Instruments. As we change how and what we teach, we must change how and what we assess. With the advent of technology, a proficient calculator user can pass many traditional exams without having an understanding of the material. The reasons given above to support various instructional methods apply to support varying assessments. Assessments can include a combination of modeling, problem solving, writing, producing or analyzing multiple representations, technology, analytic calculations, or symbolic manipulation. They can be in the form of quizzes, exams, essays, projects, problem sets, or journals. Assessments should reflect the methods and approaches the instructor has used in the classroom. Results from assessment plans that use a variety of problem types and problem presentations offer a more comprehensive view of students' understanding [5].

Developing effective assessment instruments that include various types and presentation of problems is difficult, but certainly possible. Instructors must be careful about how much they ask students to do especially on time sensitive exams or quizzes. When students are asked to explain a concept or model a situation, these requirements take longer than simply, for example, calculating a derivative using the chain rule.

Small Section Sizes. As one reads about the benefits from the possible instructional techniques above, it becomes evident many would be difficult to implement in a classroom with a large number of students. One indirect result of the methods above is that as more interactions occur between teacher and student, the teacher becomes more familiar with the students. Within smaller sections, the teacher learns more about the levels of students' conceptions, as well as their misconceptions. Assessment (not necessarily evaluation) occurs daily within the classroom activities. Additionally, in small sections teachers tend to do more of their own grading which offers a more revealing view of the students. With a manageable number of students, the teacher can then mentor these students of mathematics in an informed manner and better facilitate their learning.

Large sections of mathematics courses are certainly economically beneficial. Many students can pass through the course and the cost is only a few instructors and perhaps a few more teaching assistants. Large sections tend to enlist the assistance of graders and teaching assistants which reduces the administrative and tutoring burden on faculty. At some locations resources other than faculty and budget prescribe section sizes. These may include limited facilities and scheduling issues.

Cooperative and Collaborative Groups. Whether restricted by large section sizes or working in smaller section classrooms, cooperative and collaborative groups can assist in more interactions

between teachers and students. While both types of groups indicate students working together, some researchers and teachers make the distinction between cooperative group members having different roles, and these various players cooperate to solve problems, while collaborative group members are truly working together as one entity toward a problem solution. Group work facilitates students discussing the mathematics at hand in their own words. They must verbalize and explain their reasoning to peers. This, in turn, organizes their conceptions for improved understanding. Additionally, group and team work experiences are becoming essential for future study and employment. Group work has been successful in many settings, to include large classrooms [8] and supplemental instructional (SI) programs (see underrepresented groups) [9].

One must be cautioned that within group discussions and problem solving, misconceptions may be formulated. Feedback loops must be established in an effort to prevent this occurrence. When assigning group requirements, some members may not be as engaged as others. Although mirroring possible situations in future courses or employment that these students will have to face, some undergraduates are ill equipped to deal with the non-participatory group member. Caution should be taken when requiring a group submission. Even when all members are engaged, preparation of the final submission usually falls on one or two members of a group. Frustration with group work may disengage some students from the experience.

How to choose and integrate various instructional methods?

In choosing instructional methods one must consider both personal comfort and the environment in which they are working. The methods described above can be used intermittently as a supplemental part of a course, or daily as an integral part of a course. If we wait to be completely comfortable with a method before we try it, we might never attempt innovation. On the other hand, we should work within our comfort levels. Many times it's easier to start with small changes and then attempt larger endeavors.

The environment can have a significant impact on what one is able to do in the classroom. For example, some methods are difficult to do, although not impossible, in large sections. Time constraints on students or faculty may limit problem-solving activities. Classes at schools where most students are commuters may find it difficult to assign group projects. Technology may not be available in the form of computers, so calculators may have to suffice. Appropriate instructional methods should be chosen to fit the environment. Just as in life, any variety is bound to spice up your mathematics class.

What methods best increase success of underrepresented groups?

The largest factor in improving success of underrepresented groups may be active engagement. By using some of the techniques described above, students become engaged in learning mathematics. "Windows into mathematical thinking" are opened to previously excluded groups through alternatives to algebraic manipulation, exploration of ideas from multiple perspectives, active classroom participation and a cooperative rather than competitive environment [10]. Programs that "nurture confidence and build stronger study skills as students learn" are needed for these groups [11]. In these types of programs students have mathematical experiences that prepare them to use and understand the mathematics they have studied. A key ingredient to the effectiveness of these programs is equity of access.

At sites where reform has been adopted (or adapted), students are at least as well prepared mathematically as they have been previously and have had a more comprehensive experience [1, 12]. At some sites, programs outside of the regular classroom are offered to support all students. Support activities include mentoring and tutoring. These programs have a significant impact on underrepresented groups. One such program is Supplemental Instruction (SI), a model of learning assistance offered outside of the targeted class [9]. Perhaps these engaging environments and supplemental programs will motivate a number of students to continue their study of mathematics.

How does the learning media affect reading, writing, and problem solving?
Should calculus be a laboratory (discovery) course?

Some of the discussion to address these two questions is included in the Explorations and Discovery: Facilitated Laboratory paragraph above. The learning media may be shifting from paper to electrons, but our current students continue to need some paper support. In the future we may find this is not necessary. It is important to note that it is not the learning media itself that may affect reading, writing, and problem solving, but the way the learning media is used. Many classrooms have not made significant changes in media, but have made significant pedagogical changes that have improved reading, writing and problem solving. Those using various media feel these media are not the solution, but that they offer opportunities to improve student understanding, and hence reading, writing, and problem solving [6, 7]. One of the main advantages of varying learning media is that it allows integration of other methodologies listed above.

How to build theoretical understanding.

Understanding is a "never ending problem of consistent organization [13]." According to Piaget, two processes underlie thought and how one knows, organization and adaptation [2]. Descriptions of understanding in mathematics often focus on concepts and the skills associated with those concepts [13]. Conceptual knowledge is knowledge that is rich in relationships. These relationships are complex in nature and can be thought of as forming a web or network [5]. Procedural knowledge is knowledge that has linear relationships to other knowledge. Conceptual and procedural knowledge are critically linked and mutually beneficial for understanding. In our discussion of constructivism we assumed that students construct knowledge and that cognitive structures are always under development. In order to build understanding, instructors must arrange experiences that allow students to organize information in various ways and appropriately adapt that information to their cognitive structures. If information is presented only in a procedural manner, the student may only build linear relationships. Information presented through the use of a variety of instructional techniques in an interactive classroom facilitates the development of rich, network-like, relationships.

How to align the "achieved curriculum" and the "intended curriculum."

Three factors in aligning these two curricula are realistic expectations, communication and environment. These factors are all linked. The expectations of the intended curriculum must be communicated and understood by all those involved. What critical skills and concepts should a student who successfully completed (this would need to be defined) a particular course possess? Both follow-on course requirements and mathematics content/concept requirements would need to be considered. Additionally, these expectations would need to be realistic and aligned with the operating environment. The environment includes the facilities, equipment, faculty, time, resources, and overall goals/objectives of the school, as well as the background, needs and situation of the student population.

What guiding principles arise from educational research?

In discussing the various instructional methods, and addressing some of the topic questions, I have incorporated current educational research. The main guiding principle from educational research is

that student engagement helps students make connections among and between mathematical concepts thereby developing stronger cognitive structures and better understanding. Lynn Steen, in writing about science and mathematics education, states that although there may be disputes about content in science and mathematics courses, there is widespread agreement on several instructional principles. These include raising expectations, increasing breadth, encouraging discussion, engaging students, using computers, requiring writing, encouraging teamwork, stimulating creativity, and reducing fragmentation [14].

Summary.

This paper has presented some ideas, from the perspective of instructional practices, that may facilitate analysis of core undergraduate mathematics courses. One can see there are many choices to be made as a teacher operates along the continuum of possible levels of implementation of the various instructional methods. Although one must consider that restrictions imposed by time or resources at particular sites may prohibit the use of some of these methods, instructors should reflect on their current practices and consider possible change to instructional methods in their undergraduate courses.

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