The Impact of Technology on Experimentation and Its Implications for How and What Students Learn in Calculus

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Modern technology – including handheld technology, computers, and the World Wide Web – has dramatically expanded the kind and amount of experimentation that is feasible in the classroom and that students will use in their future careers. Virtually every student now has at his or her fingertips:

- sophisticated and compelling simulations;
- inexpensive, flexible equipment for "bench-lab" or "wet-lab" experiments;
- remotely accessible experimental equipment;
- massive and real time data sets;
- research- and museum-quality primary resources; and
- new kinds of Web-mediated distributed experimentation.

This paper discusses the implications of these possibilities on how students might best learn the subject of calculus and what topics are most important both for a first year, possibly terminal, calculus course and for a two-year core mathematics curriculum.

The two most important points we wish to make are related - the importance of modeling and the importance of experimentation and data analysis.

This paper focuses on experimentation in the various modes listed above in the context of widely available and powerful computer-based tools, including Java applets, computer algebra systems, and modeling packages. The most important point we want to make here is the importance of flexible, general purpose tools that enable students to explore ideas of their own beyond the confines of preprogrammed applets and that prepare students to do the open-ended experimentation and analysis required in the workplace. We begin with three examples.

Example 1:

Figure 1 shows our first experiment. We begin with two cylindrical tanks that are open at the top and connected at their base by a tube so that the contents of the two tanks can flow between them. The right-hand tank also has a tube or drain at its base that allows its contents to spill onto the floor. We fill the right-hand tank with water. After a short time some of this water has spilled onto the floor and some has flowed into the left-hand tank. We ask a question – **Will the water level in the left-hand tank ever be higher than the water level in the right-hand tank?**

Figures 2 and 3 show two capacitors that are connected at one end to each other and to the ground. They are connected at the other end through a resistor and the right-hand capacitor is also connected at the same end to the ground through another resistor. We charge the right-hand

capacitor. After a short time some of the charge from the right-hand cylinder has leaked into the ground and some has leaked into the left-hand capacitor. We ask a question – **will the charge on the left-hand capacitor ever be higher than the charge on the right-hand capacitor?**

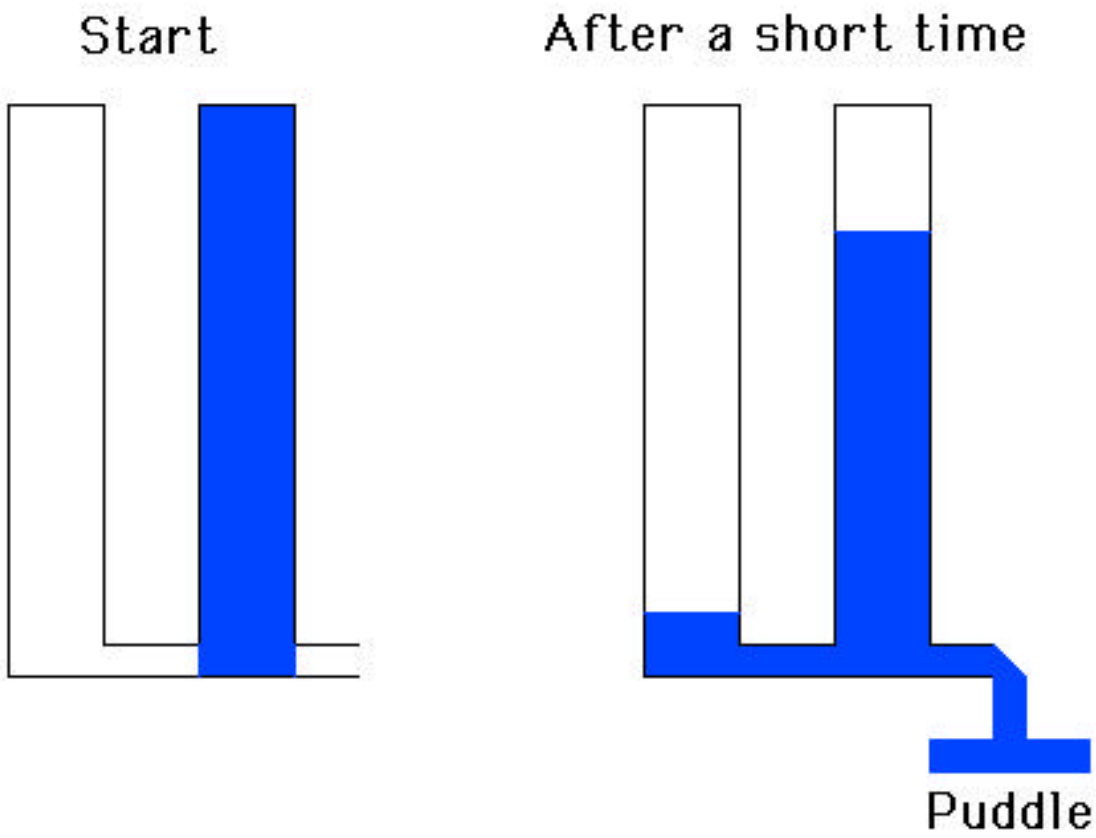

Figure 1: Two cylindrical tanks and water

You might want to pause at this point and answer this question (are they the same question?) before reading on. One of the key points of this paper is that the answer should come not from the "back of the book" but rather from experimentation. Because simulations are so cheap and common, we might be tempted to look for the answer using a computer-based simulation. Computer-based simulation, however, is not reality but rather a representation of a model that may or may not capture the essence of reality. One of our most important responsibilities as teachers of mathematics is helping our students understand the modeling behind computer-based simulations – so that they can look behind increasingly compelling, even immersive, virtual worlds to the underlying models and so that they will be able to design their own models and simulations.

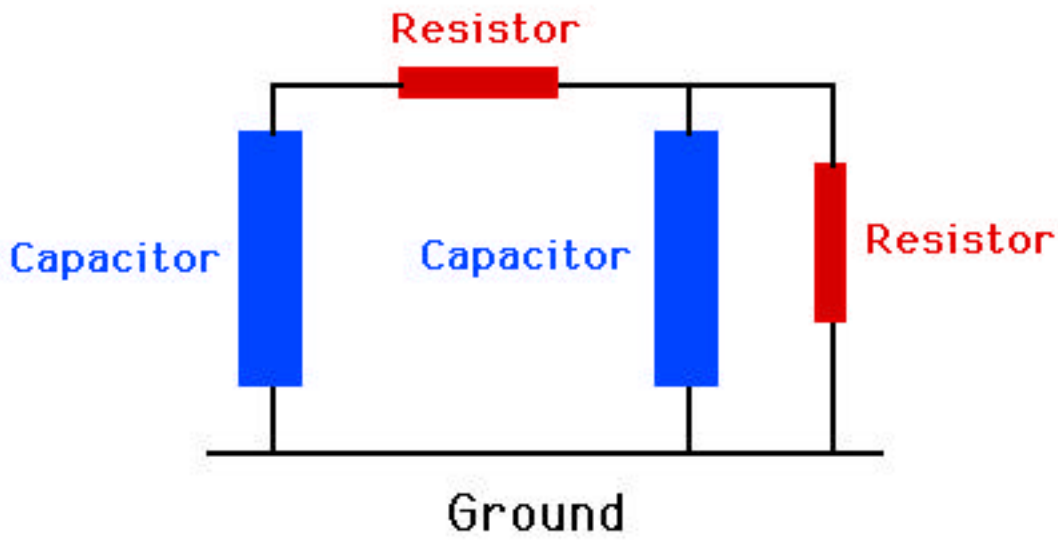

Figure 2: A schematic of an experiment with capacitors and resistors

Figure 3: An experiment with capacitors and resistors

We might model these two situations by the same system of differential equations

$$\frac{dL}{dt} = a(R - L)$$
$$\frac{dR}{dt} = b(L - R) - cR$$

where R is the charge (level of water) in the right-hand capacitor (cylinder) and L is the charge (level of water) in the left-hand capacitor (cylinder). The values of the constants a, b, and c are

all positive and depend on the physical characteristics of the components of the experiments – the diameters of the cylinders and tubes, the capacitance of the two capacitors and the resistance of the two resistors. The mathematics involved in this system of equations is accessible in a first year course:

- the meaning of the equations;
- numerical approximations to solutions;
- exact solutions; and
- qualitative analysis.

In addition, cheap and flexible equipment like the Texas Instruments Calculator-Based Laboratory puts these experiments within easy reach. Figure 4 shows the results of an experiment using the capacitor and resistor circuit above. They may be surprising.

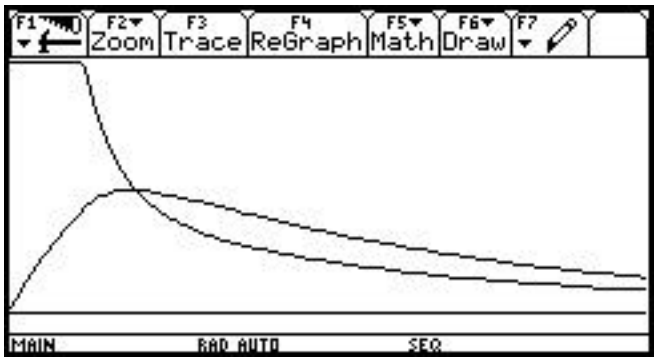

Figure 4: The results of an experiment

Notice the charge on the right-hand capacitor starts out high and drops steadily toward zero. The right-hand capacitor was charged by touching a wire from a battery to the capacitor's lead. The touch was not instantaneous – hence, the flat portion of the curve. The charge on the left-hand capacitor starts at zero; rises until it is above the charge on the right-hand capacitor and then begins to drop. Figure 5 shows a numerical approximation to one particular example of our system of differential equations.

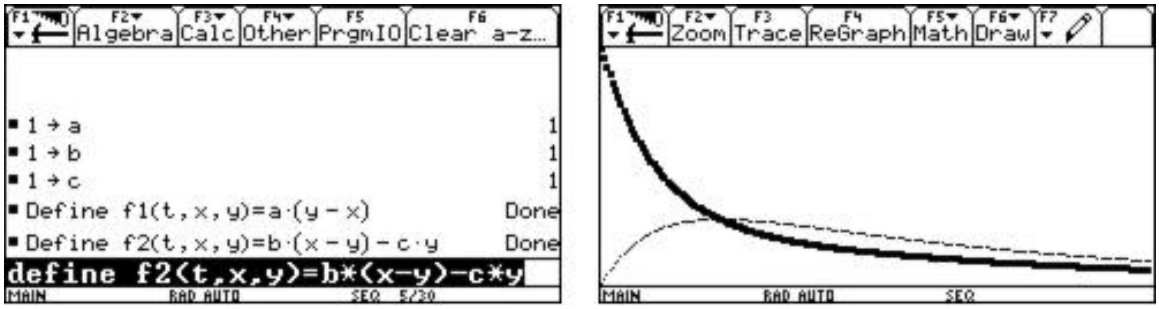

Figure 5: The results of a numerical approximation to one solution

The exact solution system of this system of differential equations is also easily accessible in a first year course. We rewrite the system of equations as

$$\frac{dL}{dt} = -aL + aR$$
$$\frac{dR}{dt} = bL - (b+c)R$$

and then the solutions are of the form

$$L = c_{11}e^{l_1 t} + c_{12}e^{l_2 t}$$
$$R = c_{21}e^{l_1 t} + c_{22}e^{l_2 t}$$

where ? ? and ? ? are eigenvalues of the matrix

$$\begin{bmatrix} -a & a \\ b & -(b+c \end{bmatrix}$$

We can obtain additional insight into this system of differential equations by looking at the variable

$$v = \frac{R}{L}$$

so,

$$v' = \frac{L\left(\frac{dR}{dt}\right) - R\left(\frac{dL}{dt}\right)}{L^2}$$
$$= \frac{Lb(L-R) - LcR - Ra(R-L)}{L^2}$$
$$= b - bv - cv - av^2 + av$$

Notice this is an interesting exercise applying the Quotient Rule.

Now we examine the differential equation or one-dimensional dynamical system

$$\frac{dv}{dt} = p(v)$$
$$p(v) = -av^2 + (a - b - c)v + b$$

The graph of $p(v)$ is U-shaped with the U-opening downward. Notice that $p(0) = b$ is positive. Thus, $p(v)$ has one positive zero, v_*; is positive for positive values of v to the left of v_*; and negative to the right of v_*. Since v is never negative, v_* is an attracting equilibrium point. Next, notice that $p(1) = -c$ is negative. Thus, v_* is less than one. This implies that as t approaches infinity v , or R/L, approaches a limit that is less than one. In other words, the level of water in the

left-hand tank (charge on the left-hand capacitor) will always eventually rise above the level of water in the right-hand tank (charge on the right-hand capacitor).

We can do better yet by using a combination of physical intuition and qualitative analysis of the original system of differential equations. This approach is actually technology-free – although graphing calculators and computer software have contributed to the growing use of qualitative analysis of differential equations. Figure 6 shows the beginning of a rough sketch showing the height of the water in the two tanks. In the beginning the level of water in the right-hand tank falls as it drains onto the floor and into the left-hand tank. The level of water in the left-hand tank rises.

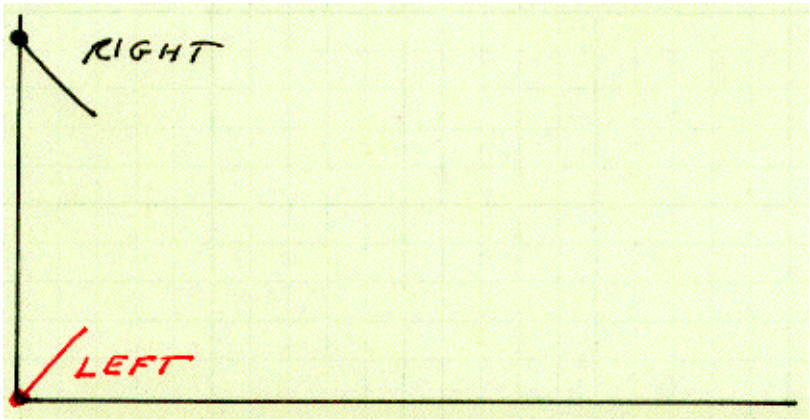

Figure 6: The beginning of a rough sketch

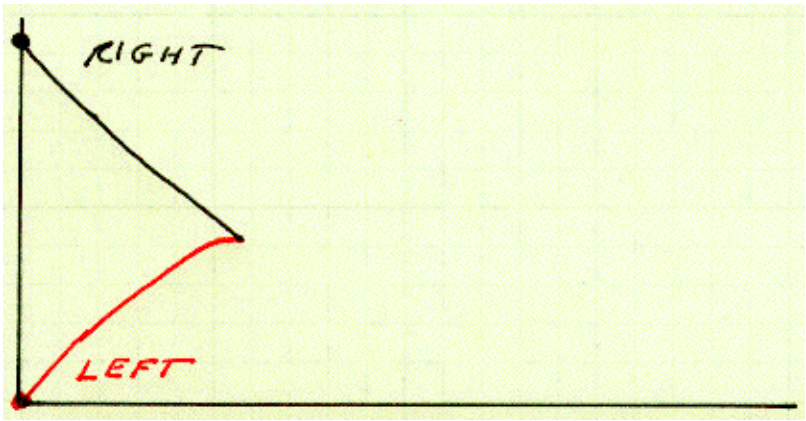

Figure 7: The rough sketch continued

Figure 7 shows what happens next. The water level in the right-hand tank continues to drop and, as long as the water level in the left-hand tank is below the level of the water in the right-hand tank, the water level in the left-hand tank continues to rise. When the water levels in the two tanks are equal, the level of water in the left-hand tank is not changing but the water level in the right-hand tank continues to drop as water drains onto the floor – that is, when the two curves cross, the curve for the left-hand tank is horizontal and the curve for the right-hand tank is still decreasing.

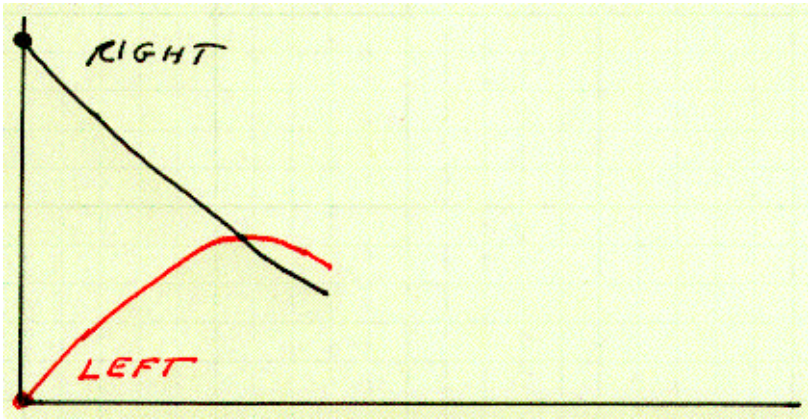

Figure 8: The rough sketch continued some more

Figure 8 continues the story. After the curves cross, the curve for the left-hand tank will decrease. Eventually all the water in both tanks will drain onto the floor.

Mathematically, our answer to the question we asked – **Will the water level in the left-hand tank ever be higher than the water level in the right-hand tank?** – is very satisfying. We did an experiment; worked out some numerical examples; solved the differential equations exactly; and gave a compelling qualitative argument – all of which were in agreement and for many people somewhat surprising. This illustrates the power and interplay of four different approaches to the question. But it still leaves a basic question unanswered – how good is our model? For example, if the tube connecting the two tanks is very wide and the tube draining the right-hand tank onto the floor is very thin, then the water might flow back-and-forth between the two tanks and our model might not be appropriate because it fails to take into account the momentum of the water.

Example 2:

This example is built around three "bench-lab" experiments using inexpensive readily available equipment, one simulation-based experiment, and investigations using a computer algebra system – all of which study interference and diffraction.

The first experiment uses a laser pointer and finely ruled slides to produce diffraction patterns. Figure 9 shows a crude diffraction pattern that was produced using "slides" made on an ordinary Postscript laser printer. These slides are downloadable from http://umastr1.math.umass.edu/~frankw/ccp/GraphPaper/diffraction/index.htm An even better slide, part of an excellent unit on the structure of DNA, is available by mail from the **Institute for Chemical Education (ICE)** at http://jchemed.chem.wisc.edu/ice/

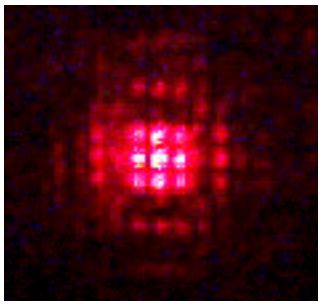

Figure 9: A crude diffraction pattern

The second bench-lab experiment uses two battery-powered speakers with a portable cassette player or a notebook computer to generate a 440 Hz tone. If the two speakers are placed outside about three feet apart and you walk along a line as shown in Figure 10 you will be able to hear the interference pattern produced. When you are at the point on the path that is closest to the speakers, their two signals will reinforce each other – constructive interference. Then, as you walk along the line, you will reach a point where the two signals are 180 degrees out of phase and almost cancel each other out – destructive interference. The underlined word "almost" is important and we will return to it. As you walk further along the line, the signals will again interfere constructively.

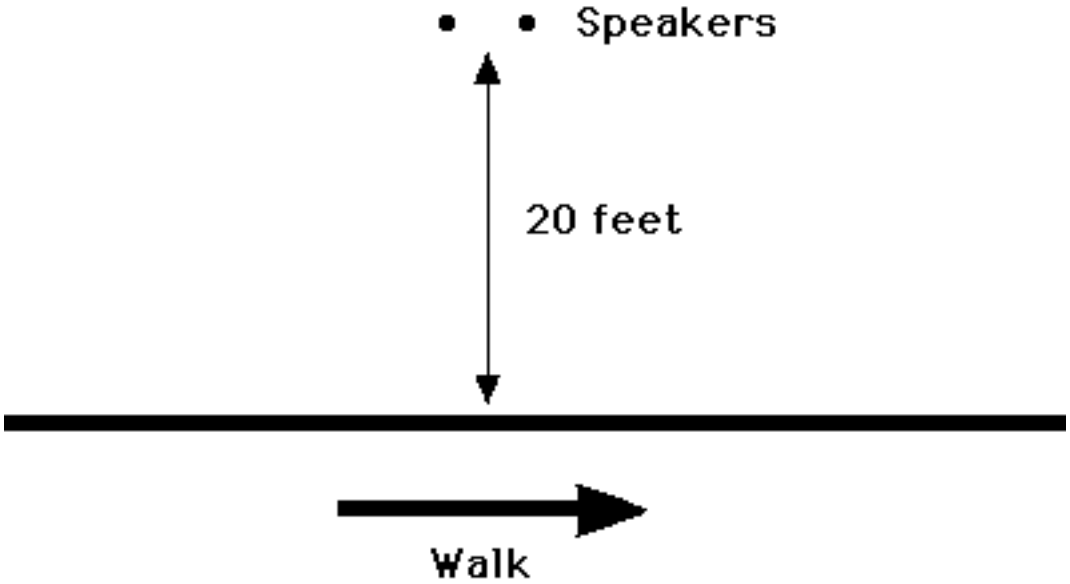

Figure 10: An experiment with sound and interference

The third bench-lab experiment uses a Texas Instruments CBL and two microphones to look at the sound signal received by two microphones at different distances from the same source. Figure 11 shows one typical result. Notice the signal recorded by the microphone that is further away lags behind the signal recorded by the closer microphone and also has lower amplitude.

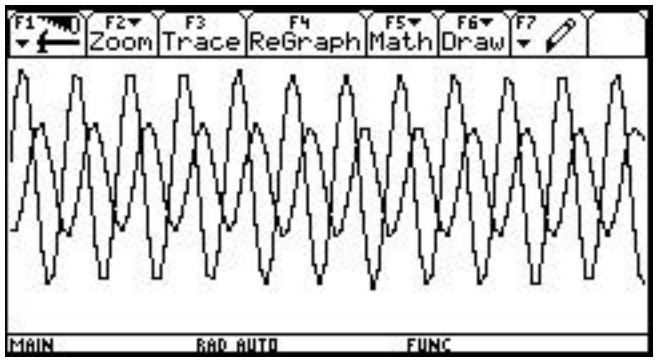

Figure 11: The sound recorded by two microphones at different distances from a source

The simulation-based experiment uses a virtual ripple tank. One Java applet is shown in Figure 12. Another is shown in Figure 13 and is available from the **WebPhysics Project** at http://webphysics.davidson.edu/Applets/Applets.html

Figure 14 shows one frame of an animated ripple tank simulation produced by a computer algebra system.

We can use these tools in various mathematics courses in different ways. One underlying message is the importance of mathematics. Mathematics is key to our understanding of lasers, acoustics, diffusion, and interference and it played a key role in the discovery of the structure of DNA. In addition to that subliminal point, we want to make two important points here – the importance of understanding modeling and the importance of general-purpose modeling tools, including computer-algebra systems.

The ripple tank applets are very impressive but they are only approximations. Neither one of them takes into account the effects of distance on amplitude. They predict complete cancellation or 100% destructive interference. Experiments with light and very finely ruled slides can be modeled effectively in this way because the distances are so large compared to the wavelength of light (physicists call this far-field interference). Our first experiment with sound, however, cannot be modeled effectively without considering the effects of distance (near field interference). After students look at the clean predictions of a ripple tank applet and then experience the less than 100% (but still impressive) destructive interference of the first experiment with sound, they can appreciate some of the subtleties of modeling. Better yet, they can use a computer algebra system to produce their own models that take into account the effects of distance. Figure 14 is one frame from an animation produced by a computer algebra system. This CAS animation is visually more impressive than either of the ripple tank applets and students can modify it to add the effects of distance.

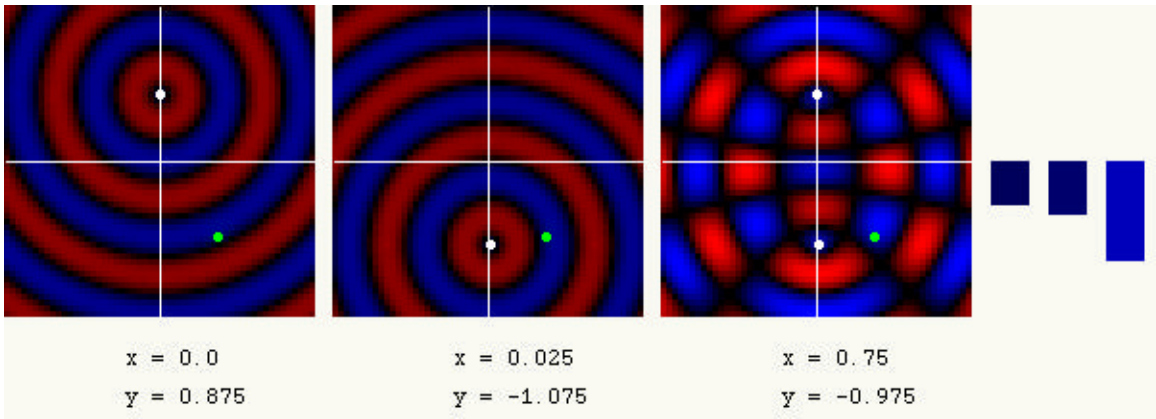

Figure 12: A ripple tank simulation

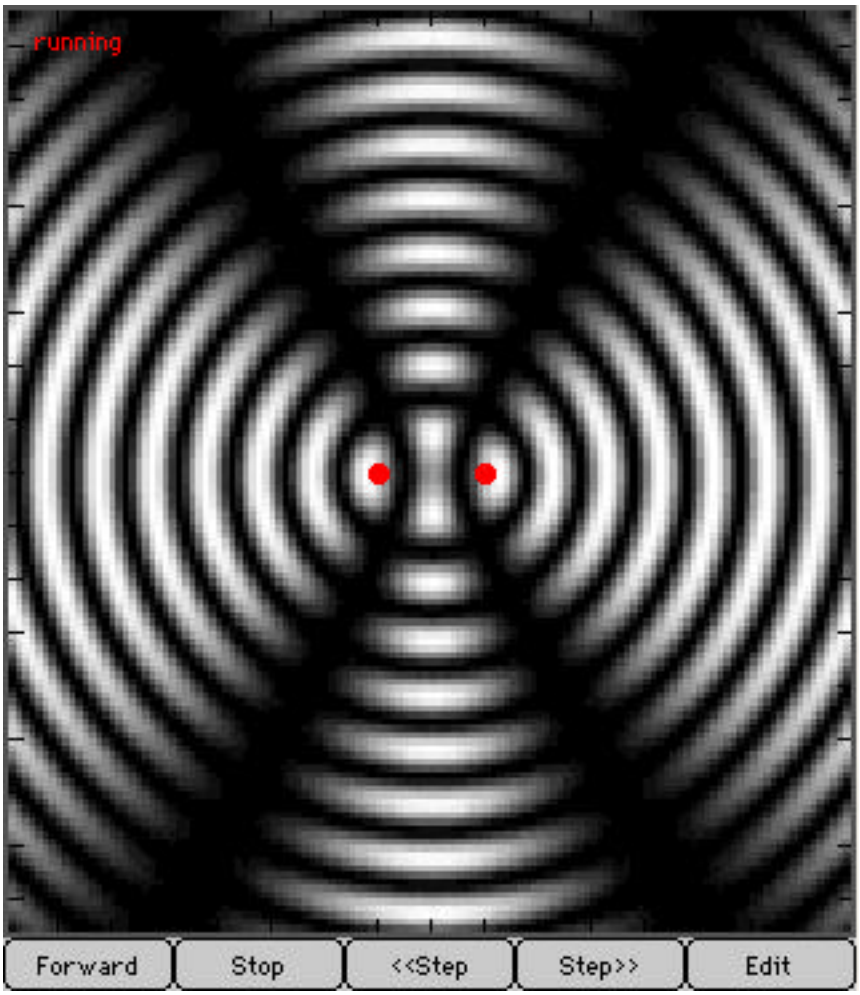

Figure 13: A second ripple tank simulation

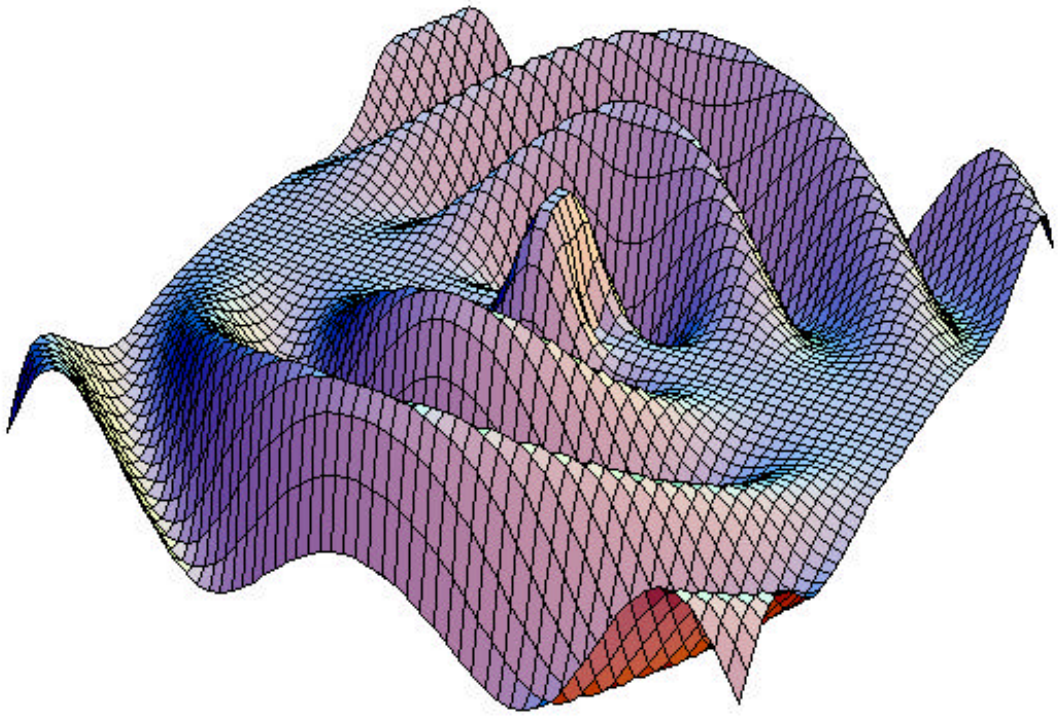

Figure 14: One frame of a CAS animation

There are some important trade-offs between dedicated simulations like the ripple tank applets and more general-purpose tools like computer algebra systems. With the former, the technology is often flashier and less intrusive. Students stay "on task" and see what we want them to see. With the latter, students must do more work and they may be distracted by the technology, but they have the flexibility to answer questions of their own and they are learning general-purpose skills.

Example 3:

Next we look at distributed collaborative experimentation. There is lots of anecdotal information about the build up of heat inside cars on hot summer days – some people crack a window; others use shades on the windshield; and others buy white cars – all in an attempt to defeat the effects of the sun. Experimentation is beyond the ability of any one person – each trial takes a full day, most of us only have access to one or two cars, and there are many uncontrollable variables. This is typical of many questions. The World Wide Web. however, together with inexpensive, portable, and flexible laboratory equipment like the Texas Instruments CBL offers some new possibilities.

- We can do massive distributed experiments in which each site records one trial, keeping track of all the environmental variables and recording the variation of temperature inside a parked car over the course of a day.
- We can share data collected by a few people who have access to lots of cars – for example, the children of car dealers might be able to do some controlled experiments using identical cars on the dealerships' lots.

These kinds of experiments are rich sources of data that can give our students a sense of ownership. Fortunately for mathematics, their interpretation raises interesting mathematical and statistical questions.

Implications:

Our students have newfound access to a startling array of experiments and experimental results. In some cases, they have access to research quality equipment, simulations, and primary data from respected sources. In other cases, the provenance is less certain and the documentation is incomplete. Even research quality data is often subject to different interpretations. In addition, students have access to professional quality tools for analysis, visualization, and model building. This cornucopia of possibilities will continue to grow as students move into their careers and as the power of technology continues its explosion.

In short, the environment in which our students will live, work, and do mathematics has dramatically changed. In this setting we look at some of the specific questions framing this workshop.

<u>Goals and Content</u>

- What are the important and difficult content choices?
- How to balance theoretical understanding with computational skills?
- How to match goals to intended audience?
- How do modeling and applications fit into the curriculum?
- What are "high standards" and how can they be achieved?
- When should calculus be taught and what other courses are needed?

Modeling, probability and statistics, differential equations, and multivariable calculus are so important that along with calculus they are an essential part of the college education of every student. Computer-based skills are enormously important. Traditional textbooks place far too much emphasis on marginally useful formal manipulation – when was the last time you had to differentiate

$$\frac{e^x \tan x^2}{\log x - 2^x}?$$

– and, strangely, omit more important formal manipulations like the Quotient Rule-based differentiation of v in Example 1. We can save some time by omitting some techniques of integration and some of the more arcane formal manipulations but I do not believe that one year is

enough for students entering without calculus. Three semesters is minimal and four would be better. The most important single factor affecting the quality of college-level mathematics learning is the time that students spend on learning and using mathematics. This implies both that more time should be spent in mathematics classes and that classes in other subjects should routinely use mathematics.

Modeling and applications are essential. We cannot be drawn into an artificial zero-sum game pitting application against theory. The synergy between them is more important than either by itself.

Realistic applications with their essential complexity impose standards that are more natural, more acceptable to students, and higher than those imposed by more traditional textbook problems.

Many people, including the author, believe that experimentation is a valuable part of mathematics courses and learning mathematics. Experimentation alone, however, is insufficient. For example, early in this paper I said, "The answer should come not from the 'back of the book' but rather from experimentation." That is certainly an overstatement. Experimentation works hand-in-hand with theory. Experimentation can show us that for certain combinations of resistors and capacitors in Example 1 the charge on the left hand capacitor does rise above the charge on the right hand capacitor but it cannot tell us, as theory can, that this always happens. But – wait – theory can only tell us that this always happens **when our model is appropriate.** Experimentation can help us test the limits of our model and develop new models when this model fails. Either experimentation or theory alone is insufficient.

Experimentation should always be guided by theory and by knowledge of the area of application. The differential equations

$$\frac{dL}{dt} = a(R - L)$$
$$\frac{dR}{dt} = b(L - R) - cR$$

that we used in Example 1, for example, did not come out of thin air but rather from a reasoned hypothesis about the behavior of the underlying physical situation. The success of this model gives us confidence in this hypothesis and increases our physical understanding.

The ease and power of technology can make a wild trial-and-error approach to modeling or an approach based on unreasoned data-fitting seem attractive. But these approaches are, at best, only useful as first steps that should be followed by a deeper understanding of the models suggested by trail-and-error or data-fitting. For example, students might observe by data-fitting that pressure and temperature are related by the equation

$$P = kT$$

where k is a positive constant and T is temperature in degrees Kelvin. Observing this relationship by itself, however, does not imply that students understand that increasing the speed of gas molecules increases both the number of times per second that a gas molecule hits a surface and

the momentum transfer that occurs with each such collision. The result is that pressure is proportional to speed squared. Understanding the relationship above follows from the further understanding that temperature in degrees Kelvin is a measure of the kinetic energy of the gas molecules which, in turn, is proportional to speed squared.

This example illustrates that the real power of technology comes not from technology by itself but rather from its use together with more traditional mathematics.

The content espoused in this paper is important for everyone – both math majors and anyone who may need to make decisions based on quantitative analysis and mathematical models. It is most particularly important for those students who will become our future college professors and teachers and who will be responsible for the education of our children.

Technology

- How should technology affect how and what we teach?
- What are the strengths/drawbacks of different technology choices?
- How to match technology choices to the intended audience?
- What are the possible effects of future technology?

The first question has been addressed above. One aspect – the importance of using both general-purpose tools like CAS and computer programming and dedicated tools like Java applets – of the second question has also been addressed. The impact of both current and future technology depends more on how we use it than on the technology itself. As educators whose first concern is that all our students develop the ability to use mathematics, we must insist that the quality of learning and its accessibility are the driving forces as we make choices about technology. Our scarcest resource is time and our second scarcest resource is money. We should invest in core capabilities – hardware that is capable but not extravagant and software and operating systems that are reliable and do not squander resources.

Areas to watch for the future impact of technology on education –

- Content – curriculum materials that use technology to improve learning. The value of a television set depends primarily on the programs and videotapes we watch. Authorware and mechanisms supporting live, interactive mathematical content are particularly important.
- Information access – three cheers for the World Wide Web!!!
- Tools – for analysis and visualization.
- Collaboration – mechanisms that enable us to work together using shared resources and tools.
- Reuse and sharing of work. Software buzzwords like “granularity” and “components” can support distributed and collective authorship. We need new models and practices for intellectual property rights. Reuse is the key to progress instead of waste, to reliability and quality, and to stability as the underlying platforms continue to evolve. The culture of curriculum development should adapt some of the practices of research – building on each others’ work should become the norm rather than the exception.
- Simulation – immersive simulations, virtual reality and even virtual worlds.

- Bench labs and wet labs.

> Perhaps, most importantly, the real impact of technology on our world will depend on whether we use technology to extend our ability to work collaboratively and to make informed, thoughtful, and ethical choices as individuals and as a society.