Towards a Framework for Interdisciplinarity

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INTRODUCTION

Linear algebra, ordinary differential equations, and multi-variable calculus hold an important position collectively within the standard second-year curriculum for many undergraduate science, mathematics, engineering, and technology (SMET) students, particularly those in engineering and the physical and applied mathematical sciences. Although many natural interconnections exist among these three subjects – with ideas and techniques from each in constant use in the others – their traditional organization and "delivery" as distinct courses can create artificial barriers among the subjects in the minds of the students. *If we are to break down boundaries between disciplines for student learning, perhaps we should first break down subject boundaries within our own discipline.* In this paper we explore an organizing framework consisting of two primary themes and several supporting conceptual axes, which we believe can promote an interdisciplinary approach to teaching and learning the central ideas from these three subjects.

CURRENT SETTING

Recognizing the centrality of the three subjects mentioned above, the importance of using motivating applications, and the opportunities for new pedagogical approaches afforded by advances in computational technology, the mathematical sciences community has been engaged over a number of years in various efforts to improve the teaching of these courses and student learning of this subject matter. For example, faculty enhancement workshops such as *Computer Aided Instruction in Linear Algebra and Ordinary Differential Equations* in 1990 [29] and the **ATLAST** [14] and **CODE-E** consortia [6] both begun in 1992, have engaged large numbers of faculty across the country in rethinking their pedagogical approaches. More recently, a number of curriculum development projects have produced new teaching and learning materials for these three subjects. These include a MathCad based laboratory approach to linear algebra by Porter and Hill [22], the *Internet Differential Equations Activities* project [15] and the text from the *BU Dynamical Systems* project [4] in differential equations, and the **Calculus Consortium at Harvard's** text *Multivariable Calculus* [16] and Cheung's work at Boston College with *Maple* and multivariable calculus [7]. Some of these efforts have led in turn to "second generation" faculty enhancement workshops and conferences as part of the dissemination work of these various projects. Additional projects and innovations are listed in the references.

While much progress has been made, several important challenges still exist, which argue for a *conceptual and contextual unification of subject matter*. Firstly, depending on a student's major requirements, the sequence in which students enroll in these courses can be quite varied; and institutions differ greatly in the frequency and the order in which these courses are offered. Versions of these courses are also taught by other departments for their own majors. For example, vector calculus courses for engineers are often taught under the name "Engineering Analysis". While this cafeteria plan of courses may maximize choice, it also encourages students' conceptual learning to remain disconnected. Subject matter is "covered" repetitively, but not <u>revisited</u> in a coherent way; and opportunities to take advantage of and reinforce the synergistic interconnections

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among these three important mathematical subjects are lost. Secondly, opportunities to make explicit linkages with application areas encountered by students in their major-specific courses are not generally exploited. Often these are courses in which they are either concurrently enrolled as second-year students or will be enrolled as third-year students, e.g. electromagnetism, statics, circuit theory, or systems and control. This practice of fragmentation ignores the larger curricular context in which the courses are being offered. Finally, while not a universal situation, the fact remains that many students take these courses in large enrollment settings which have generally not proven to be conducive to optimizing student learning[2]. In this paper we will not address explicitly the issues involved in restructuring student learning environments (see for example [5,8,9,21]). However, the subject is a rich and important one on which many continue to work[3].

CONCEPTUAL FRAMEWORK

"All exact science is dominated by the idea of approximation."
- Bertrand Russell[4]

We propose a year-long integrated course of study emphasizing two themes: 1) linear models in their own right and 2) linearity as a tool to help understand non-linear phenomena (the process of linearization). These dual themes would offer a way to unify the "standard topics" contained in the typical three-course collection of linear algebra, differential equations, and multivariable calculus, and to promote the connected learning of their core ideas. While the total number of "contact hours" for a course of study of this type would be the same as the three separate courses combined, we believe that the *whole can be greater than the sum of its parts*. Typically, for any given unit of time within a term, students are constrained to an equal distribution of formal classroom time in each of the three courses (or worse, only one or two of the courses with the others taken in another term!). Variable sequencing of the courses further contributes to creating a set of disconnected learning experiences.

The model we propose seeks to break out of this administrative confine, by creating a learning structure that embraces the rich interconnectedness of these three subject areas and enables students to learn the core ideas in a <u>concurrent</u> manner. Each of the three single-term courses typically carries three or four credits, with four to five contact hours each week. In place of this structure consider one in which there are six contact hours weekly throughout a year-long (two semester) sequence. To maximize active student engagement with concepts through a laboratory learning environment, a combination of one- and two-hour class blocks totaling six hours could be scheduled weekly. (Institutions with different credit/contact hour figures, could consider analogous restructuring.) In any given week of a term, the relative amount of "coverage" of material usually identified with linear algebra, ordinary differential equations, or multivariable calculus can vary, and this distribution can change from week to week. At the end of such a course of study, we believe the cumulative effect can be at least equivalent to that of an existing system in terms of mere "coverage" of standard topics, but <u>much greater</u> in terms of student learning, as a result of the coordinated context in which the learning takes place.

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[2] Characteristic of these settings is the concatenation of large-lecture sections taught by faculty with smaller recitation sections taught by graduate students or adjuncts. Students face different instructional approaches and styles in this practice, and are also often with different classmates depending on the learning venue, further fragmenting their learning experience.

[3] The integration of formal lecture sections, recitation and/or discussion sections into a "studio" or laboratory setting coordinates and reinforces opportunities for student learning and understanding. In recent years, RPI has developed successful studio approaches to teaching and learning, e.g. Studio Physics [28] and Studio Calculus [10,11].

[4] B. Russell, "The Scientific Outlook", p. 63, *The Free Press*, Glencoe, IL, 1931.

Several projects are in fact working towards this vision, for example, the *Linearity I and II* project (Black et al, see [3]) is experimenting with the structural changes and the use of "mini-projects" from science and engineering, while the *Coordinated Curriculum Library* project (Moore et al, see [20]) and the *Connected Curriculum* project (Wattenberg et al, see [27]) are developing materials and laboratory modules.

Supporting the dual themes of linear models and the linearization process, we envision several axes which represent different modeling perspectives which collectively define a conceptual space into which projects, experiments, and other learning materials and tools can be placed. One axis consists of a one-dimensional to multi-dimensional perspective, a second axis represents the interplay between discrete and continuous models, and the third axis captures the contrast between deterministic and stochastic phenomena and/or assumptions.

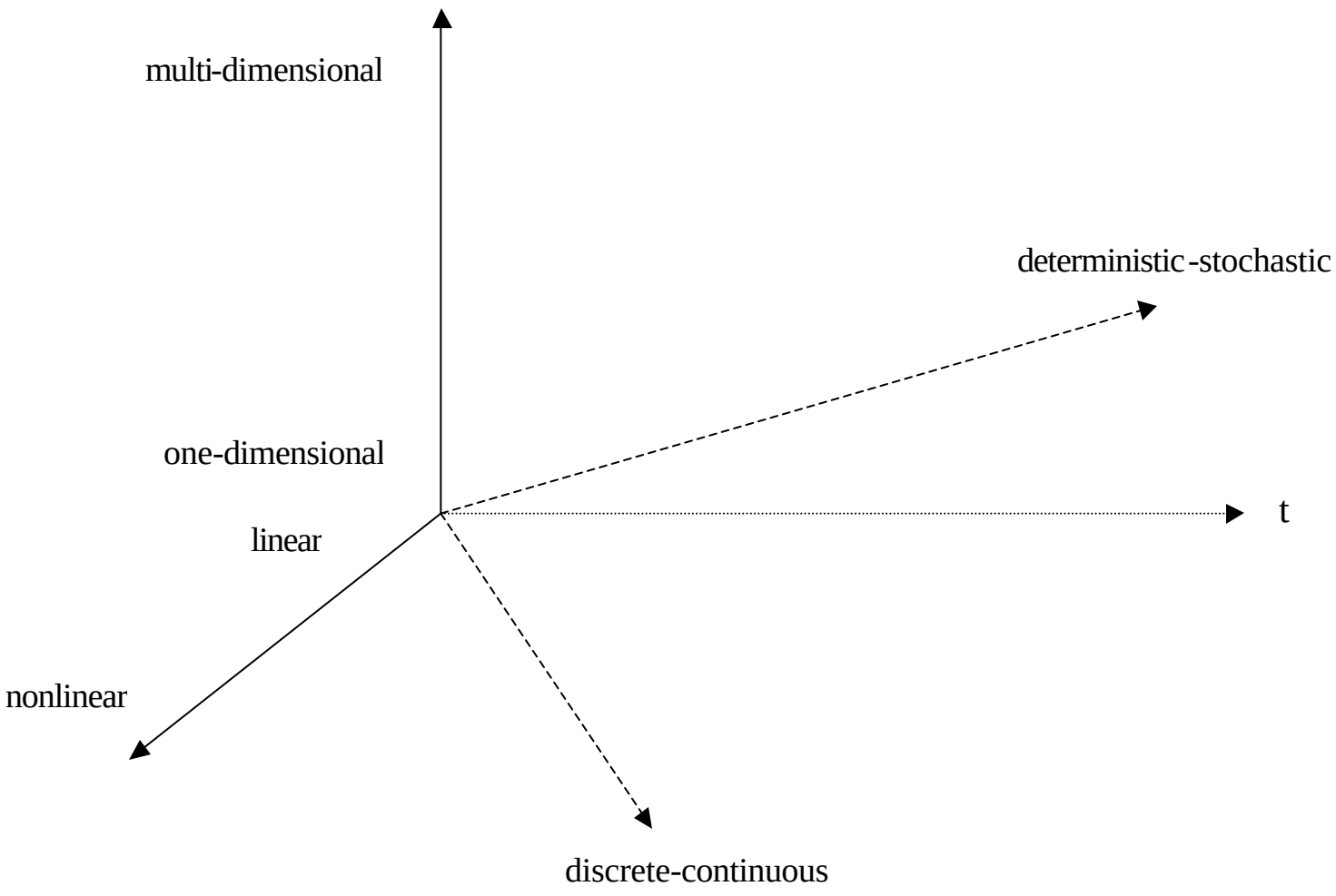

Figure 1: A conceptual framework

A PROTOTYPE SCOPE AND SEQUENCE

If the first-year curriculum in mathematics aims to develop a rich collection of "ways of knowing" a single valued function of one variable, then the second-year should develop a similar understanding for single- and vector-valued functions of several variables. The most fundamental of these functions are the linear ones, and the presence of multiple variables (both input and output) argues naturally for taking up linear systems from the outset. Thus, we envision the year sequence beginning with linear systems of equations and developing the basic language of linear algebra: vectors, vector operations, superposition, etc. While in the first two or three weeks the distribution of "coverage of standard topics" would be weighted significantly towards linear algebra, this emphasis could be balanced by an early introduction to the second major theme of the course: the use of linear approximations. In both cases these introductions can be naturally connected to and complemented by realistic laboratory experiments, either brought directly into the mathematics classroom or investigated concurrently in allied departments. For example, truss problems (described below in greater detail) provide a natural way to discuss linear systems of equations. Likewise, initial investigations of surfaces and level curves, partial derivatives and gradients, and tangent plane approximations could be linked to electrostatics, where students could observe and construct equipotential lines on a charged plate. Elementary fluid flow examples could also provide useful context.

In our hypothetical model, the first term of the sequence would deal primarily with static models of science and engineering phenomena which find expression as purely algebraic equations. A particularly interesting and value-added feature afforded by this approach is that the data fitting issues that arise naturally in analyzing experimental observations lead directly to least squares problems. This of course leads to orthogonality, Gram-Schmidt orthogonalization, and the QR algorithm. In addition, the groundwork is laid for more advanced model validation issues encountered in upper-division courses. During the second term of the sequence, we see the program of study changing to one largely dominated by consideration of dynamic models and science and engineering phenomena which express themselves through dynamical systems of equations. It would be especially valuable to consider examples that begin as a static model, and can later be revisited as a dynamic one.

Within the framework depicted in Figure 1, it is possible to bring out the important interplay between the arithmetic and algebra of discrete observations and the calculus of the infinite. For example, a natural introduction to dynamical phenomena and models is provided by examination of difference equations $x_{n+1} = Ax_n$. This approach capitalizes on the early grounding in the language of linear systems we propose. Eigenvalue-eigenvector analysis can be motivated by numerical computational experiments and visual representation of the iterations, which reveal the dominant eigenvalue and associated eigendirection[5]. The concepts of equilibria and stability arise naturally along with the use of linear approximations of the multivariable functions that define the vector fields. Subsequently these ideas are revisited in the form of differential equations and dynamic (time-varying) models of science and engineering applications.

PROMOTING CONTEXTUAL LEARNING

We believe the conceptual framework described above can provide multiple entry points from which interaction with science and engineering colleagues can proceed to establish and reinforce connections with concurrent, core disciplinary courses such as statics, linear circuit theory,

[5] This is of course the basic idea behind power methods for numerical eigenvalue calculations.

dynamics, structures, electricity and magnetism, fluid mechanics, and systems and control. Indeed, there is emerging evidence (see for example [17-19,23]) that conceptual learning can be positively impacted by the use of physical models and "hands on" experiments. While implementation of these cross-disciplinary interactions can take the form of guest lectures, demonstrations, and shared use of equipment, we believe it is worth considering the systematic and systemic incorporation of such experiment-based learning tools directly into the mathematics classroom. For the past several years innovative faculty have begun to take advantage of the ready availability of simple hand-held Calculator Based Laboratory (CBL) devices from Texas Instruments, and more recently Palm Pilot platforms [26] have even begun to make their way into middle school classrooms. We describe several examples that are chosen to help illustrate core mathematical ideas and provide opportunities to make explicit connections to concepts that students encounter in their courses used to satisfy major requirements.

Structures in equilibrium. Planar and three-dimesional trusses are considered, both constructed for illustrative purposes and taken from the real-world[6]. The related subject matter in mechanical engineering and civil engineering as well is *statics*. Strain gauges record the tension in the frame members as a result of loads applied at different nodes of the structure. A linear system of equation $\boldsymbol{Ax=b}$ relates the vector of displacements in the frame members to the vector of applied forces at the nodes. The matrix $\boldsymbol{A}$ is known as the stiffness matrix.

This case provides a concrete example of a linear system of equations and an introduction to Gaussian elimination as the method of solution. Linearity or the principle of superposition can also be "experienced" first-hand by students as they observe the effect of applying different vectors of loads and measuring that the "output of the sum (of inputs) is the sum of the outputs (from each input)". The $\boldsymbol{LU}$ decomposition[7] that records the steps of Gaussian elimination is motivated by the problem of needing to determine the responses of the truss due to multiple loading vectors (different right-hand side vectors $\boldsymbol{b}$). A particularly attractive feature of this example is that the two methods of analysis used by engineers, the so-called method of joints and the method of sections are in fact mathematically dual formulations of the problem.

Many points of departure are possible from this example including the investigation of non-linear stress-strain relationships and a subsequent linearization to understand local behavior, and the investigation of ill-conditioning via consideration of a nearly statically indeterminate system. The latter investigation has particularly importance, since the increased use of powerful software packages for numerical computation argues for increased attention to understanding issues of numerical stability.

Analysis of electric circuits. Basic examples of linear electric circuits are certainly an important component of the electrical engineering and physics curriculum. The connectivity matrix (or edge-node incidence matrix) describes the topology of the electrical network. Kirchoff's Laws and Ohm's Law combine to yield linear systems of equations relating potentials and currents to applied voltages and current sources. Measurement devices can allow students to observe first-hand the linear input/output relations embodied in these circuits.

[6] For example, members of the Department of Mechanical Engineering at Rennselaer Polytechnic Institute have used bicycle frames for similar purposes.
[7] The more general matrix factorization is $\boldsymbol{PA=LU}$ to accomodate row interchanges.

As in the preceding example, this case presents another source of "real-world" matrices and linear systems of equations[8]. In fact, Kirchoff's Voltage and Current Laws find equivalent mathematical expressions in terms of descriptions of the column space of the connectivity matrix and its left nullspace, respectively. Key mathematical concepts such as orthogonality and the interrelationship of the four fundamental subspaces associated with a matrix (in addition to the two just mentioned, there are the nullspace and row space) arise concretely in terms of the physical context, see [25] for example. Furthermore, a discussion of linear independence and basis can be made explicit through reference to loops in the graph and the number that are independent. In terms of the discrete-continous conceptual axis alluded to above, the edge-node incidence matrix can be viewed as a discrete approximation to the derivative operator. Then the fact that its nullspace consists of the one-dimensional subspace of constant vectors coincides with anti-differentiation only being determined up to the set of constant functions.

Again, there are many points of departure for subsequent revistation afforded by use of this case. Introduction of capacitors and inductors to the electric circuit moves the problem into the realm of dynamic or time-varying models, i.e. differential equations. But preserving the fundamental linear nature of the problem yields another opportunity to encounter the study of eigenvalues and eigenvectors. Further on, nonlinear circuits could be considered, for example students could experiment with a van der Pol oscillator, allowing motivation for the qualitative analysis of the phase plane. The particular consideration of a van der Pol oscillator would also allow investigation of bifurcation phenomenon and an opportunity to motivate the use of linearization to gain insight into local system dynamics.

Oscillations and periodic phenomena. An air track with a collection of masses and spring is the physical system considered. This application is representative of subject matter in mechanical engineering, electrical engineering, and physics relating to the broad area of *dynamics*. Newton's Second Law and Hooke's Law combine to yield a second-order linear system of differential equations $\frac{d^2x}{dt^2} = Ax$.

This example again provides a concrete example of a linear system of equations. For a two-mass, three-spring system (with equal masses and equal spring constants), the corresponding matrix $\boldsymbol{A} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ has eigenvalues which yield the frequencies of oscillations of the so-called "fast" and "slow" modes of oscillation[9]. Furthermore the eigenvectors of $\boldsymbol{A}$ correspond to the two initial conditions (the vector of initial displacements, with zero velocity) that produce exactly these fundamental modes of oscillation. One mode begins with the two masses displaced an equal amount in the same direction, ***x(0)=[1,1]*** and the second mode begins with the two masses displaced equal amounts in the opposite direction, ***x(0)=[1,-1]***[10]. For further investigation, students

[8] Changing applied voltages in these circuits is analogous to changing applied forces in the truss problems, with both problems providing additional entry points to a discussion of stability of matrix calculuations and conditioning.

[9] In fact the frequencies are the square roots of $-l_j$, the negative of the eigenvalues.

[10] The author has used this experimental device numerous times in both linear algebra and ordinary differential equations courses. Students count the number of oscillations for the different modes in a ten-second interval and then compare their ratio. This procedure usually yields one decimal place accuracy to the theoretical value, much to the surprise and pleasure of the students. Students also report overwhelmingly that such an experiment contributes to their understanding and helps connect their mathematics courses to their other science and engineering courses.

could increase the number of masses and springs, consider the effect of different masses and different spring constants (qualitative dependence on parameters), include frictional effects in their model, and consider non-linear effects as well. A particularly interesting avenue to follow is to consider the limiting case of an infinite number of masses and springs (each of smaller length), a model which could lead to a discussion of the wave equation and conservation principles.

Fluid behavior. Experimental apparatus can also be constructed to illustrate fluid mechanics concepts which are particularly germane to students in mechanical engineering and physics. The Hele-Shaw cell consists of two clear plastic plates separated by a thin space into which a fluid may be injected through various input holes. The narrow spacing confines the fluid to be essentially two-dimensional. Here we envision that the use of a real physical fluid can help illustrate a vector field and motivate concepts such as its divergence and curl. To that end, students could inject colored dye into the fluid to help mark the flow field, or alternatively, small particles such as aluminum chips can be used. If the fluid velocity is slow enough, actual measurements can be taken. Incompressibility can be felt and mathematically verified **div v** = 0.

CONCLUSION

By *reorganizing mathematical content* with an emphasis on contextual learning, the model we propose seeks to take advantage of the natural synergy of these three core mathematical subjects and their place within the broader context of second-year SMET education. In combination with a *restructured learning environment*, this approach can offer a vehicle for bringing thematic and pedagogical coherence to much of the second-year SMET curriculum. This in turn provides a unified framework for learning that is applicable to a significant number of SMET students. While we believe that many institutions will find value in this approach, it may hold special appeal to institutions that enroll large numbers of engineering and/or applied science students. In particular, the "coherency and efficiency of coverage" afforded by the framework may help address an overcrowded engineering curriculum. For engineering schools in particular this approach is consistent with new ABET certification requirements. It is also interesting to speculate about the application of this framework to linear algebra, multi-variable calculus and probability and statistics in support of a curriculum that reflects a focus on environmental/civil engineering, chemical engineering, and earth science.

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