Problem of the Week, Fall 2006 16 November 2006

Circle One: Cadet Faculty Student (non USMA) Faculty (non USMA) Other

Last Name (Please Print):

Problem 11

Let n be an integer. Prove that if $2\sqrt{28n^2+1}+2$ is an integer, then it is a perfect square.

Since $2\sqrt{28n^2+1}+2$ is an integer, $\sqrt{28n^2+1}$ is an integer.

Since $\sqrt{28n^2+1}$ is an integer, $(28n^2+1) = (2m+1)^2$ for some non-negative integer m

$$28n^2+\cancel{1} = (2m+1)^2 = 4m^2+4m+1$$

$$= 4m^2+4m+\cancel{1}$$

& 1's cancel & divide both sides by 4 gives us

$$7n^2 = m^2+m = m(m+1)$$

Since m & $m+1$ are relatively prime,

either $[m = 7s^2 \ \& \ m+1 = t^2]$ or $[m = u^2 \ \& \ m+1 = 7v^2]$

Since the residue of $u^2 \bmod 7$ can only be 0, 1, 2, 4,

$7v^2 - u^2 \geq (7-4) = 3$ & the latter is not a possibility (i.e. $7v^2 - u^2 = 1 \neq 3$)

$\therefore \ m+1 = t^2$

$$\& \ 2\sqrt{28n^2+1}+2 = 2(2m+1)+2$$

$$= 2(t^2+t^2-1)+2$$

$$= 4t^2$$

$$= (2t)^2$$