Problem 2: Black Night

USMA D/Math Problem of the Week
Submission Deadline: September 27, 2007 at 1600
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Problem Statement: Three treasure hunters are traversing a jungle in the dead of night looking for a jewel-encrusted box filled with gold. They split up, and at midnight arrive at independent random positions along an old road represented by the interval x 2 [0, 2000ft]. They don't know it, but the box is sitting on the road at precisely x = 1800ft. With flashlights, they can see up to 200ft along the very straight, flat road.
(a) What is the probability that one of them can see the box?
(b) The box is too big to carry alone, and shouts for help travel only 300ft in the night. What is the probability that the box will be found and someone will be in range to come help carry it away? [Assume the hunters do not move along the road.]

NOTE: I answered this prior to seeing the clarification. Part a should be fine, I made an assumption in b that may or may not follow the intent of the clarification, but I think is a valid way of seeing the problem.

a) A hunter can see the box if his/her x position is between 1600 and 2000. Since there is an equal (uniform) probability of being anywhere from 0 to 2000, there is a (2000-1600)/2000 = 0.20 probability of any hunter being able to see the box. The probability of at least one hunter being able to see the box is simply 1 – P(none can see box) which is $1 - (0.80)^3 = 61/125 = 0.488$.

   Using a binomial distribution (n = 3 identical trials, success or failure outcomes, probability of success is constant at p = 0.20, trials are independent, random variable of interest is the number of successes of the 3 trials) with n = 3 and p = 0.20, it is easily determined that the probability no hunters will find the box is p(0) = 64/125 = 0.512, the probability that (exactly) one hunter will find the box is p(1) = 48/125 = 0.384, the probability of two hunters finding the box is p(2) = 12/125 = 0.096, and the probability that all three hunters will find the box is p(3) = 1/125 = 0.008.

b) I assume that if 2 or 3 hunters are within 200 feet of the box, they will both see it and be able to carry it together, regardless of the hunters' relative positions (two could be more than 300 feet from each other, but both still see the box). The probability of more than one finding the box is p(2) + p(3) = 12/125 + 1/125 = 13/125 = 0.104.

   We also need to account for the possibility of 1 hunter finding the box (1600 <= x <= 2000) and one (or both) of the others being within 300 ft of the one who found it. Given that exactly one hunter found the box, the other 2 hunters' positions are random (bivariate uniform) between 0 and 1600 (for each hunter). If both hunters are between 0 and 1300, it is impossible for either to hear the call of the box finder, regardless of the finder's position.

   If either (or both) hunters are between 1300 and 1600, it is possible, but not a certainty that one (or both) will hear the finder, depending on the position of the finder, which is random (uniform) between 1600 and 2000. For instance, if the greatest position of the 2 hunters that didn't find the box is 1400, the finder must have a position between 1600 and 1700 in order to be heard by one of the others. This will happen with probability (1700 – 1600)/(2000 – 1600) = 0.25. If the greatest position of the 2 hunters that didn't find the box is 1500, the finder must have a position between 1600 and 1800 in order to be heard by one of the others. This will happen with probability (1800 – 1600)/(2000 – 1600) = 0.50. If the greatest position of the 2 hunters that didn't find the box is 1600, the finder must have a position between 1600 and 1900 in order to be heard by one of the others. This will happen with probability (1900 – 1600)/(2000 – 1600) = 0.75. Clearly, the probability increases linearly from 0 to 0.75 as the position of the non-finder increases from 1300 to 1600. (And the hunter with the greatest x position is the one of interest, making the surface of probabilities symmetric along $x_1 = x_2$.)

A double integral across all x_1, x_2 positions (positions of the non-finders) that have a non-zero probability of hearing the finder can be created to determine the probability that someone is in hearing range given that exactly one hunter found the box:

$$\int_{x_2=13}^{x_2=16}\int_{x_1=0}^{x_1=x_2}\frac{1}{256}\left(\frac{1}{4}x_2-\frac{13}{4}\right)dx_1dx_2+\int_{x_1=13}^{x_1=16}\int_{x_2=0}^{x_2=x_1}\frac{1}{256}\left(\frac{1}{4}x_1-\frac{13}{4}\right)dx_2dx_1=\frac{270}{2048}$$

In order to find the probability that exactly one hunter found the box AND (at least) one of the other hunters is within earshot, we simply need to multiply the probability that exactly one hunter found the box and the probability that (at least) one of the other hunters is within earshot given that exactly one hunter found the box:

$$\frac{48}{125}\cdot\frac{270}{2048}=\frac{81}{1600}=0.050625$$

Accounting for the possibility of two or more hunters finding the box and the possibility of exactly one hunter finding the box with at least one other hunter being close enough to hear, we have a total probability that the box will be found and someone will be in range to come help carry it away of:

$$\frac{12}{125}+\frac{1}{125}+\frac{81}{1600}=\frac{1237}{8000}=0.154625$$