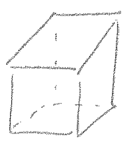
Problem 5: The Firetruck

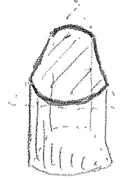
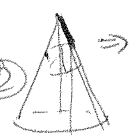
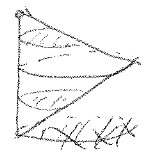
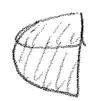
USMA D/Math Problem of the Week

Problem Statement: Toddler toys often come equipped with holes and blocks in various shapes. I recently purchased a toy firetruck for my son with three holes: a 2x2in square, a 2in diameter circle, and an isosceles triangle with base and height both 2in. What is the volume of the *largest possible* solid which (i) can be cut from a 2in cube, and (ii) will pass through all three holes?

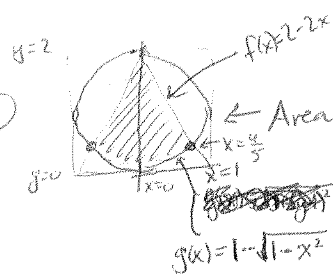


A cube has volume $8\,in^3$. To pass through the circle, it must have the form of a cylinder  which has volume $2 \cdot \pi \approx 6.28\,in^3$.

Assume that the "triangle pass-through" occurs along an axis of the square. Then, the block looks either like

①  or ②  ⇒ sections  or  ⇒ sections 

Option ①

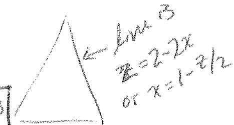


$\leftarrow \text{Area} = 2\int_{x=0}^{4/5}\left[(2-2x)-(1-\sqrt{1-x^2})\right]dx = 2(.863)$

⇒ Total Volume $4 \times .863 = \boxed{3.44\,in^3}$

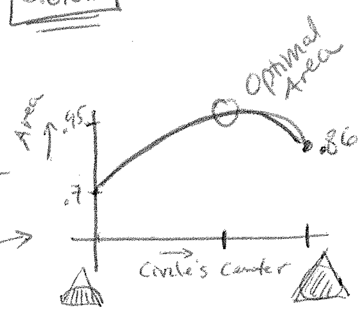
Option ② A Typical cross section  has area $\int_{x=x_0}^{x=x_0} 2\sqrt{1-x^2}\,dx = 4\int_{x=0}^{x_0}\sqrt{1-x^2}\,dx$

So the total volume is $4\int_{z=0}^{2}\int_{x=0}^{1-z/2}\sqrt{1-x^2}\,dx\,dz = 4(.904) = \boxed{3.616\,in^3}$



Option ③ will be smaller... Check and see!

Option ④ There's a better 4th Option though: the cylinder with cross section  → Area Curve



It's max. area is $\approx 1.89\,in^2$, so the max volume is $\approx \boxed{3.78\,in^3}$

RMK: ~~I don't have a proof that this is the correct answer, but it seems likely.~~