Problem 15: Concentric Chase

USMA D/Math Problem of the Week - AY2008

Submission Deadline: March 6, 2008 at 1600

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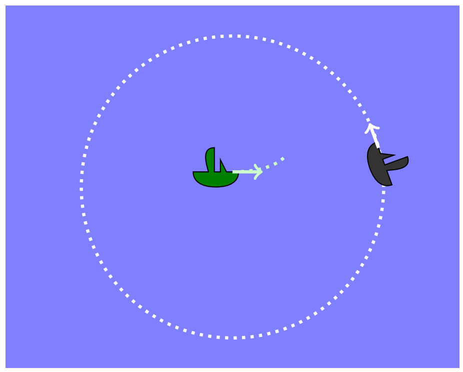
Problem Statement:

Two boats are traveling in open water in circular patterns. One boat is traveling in a circle along the parametric curve $\mathbf{r}(t) = \langle 10\cos(\pi t), 10\sin(\pi t)\rangle$ (constant speed 31.41592653589793 knots). The second boat can travel at half that speed, and *always heads exactly towards the other boat.* Find the location of the two boats when $t = 200$.



(Note: assuming that the two boats are relatively close to begin with, you do not need to know the starting location of the second boat. A numerical solution is okay, but an answer will only receive full credit with some sort of analysis.) *This problem was inspired by my 1-year old son and a remote control Hummer.*

Solution:

We begin by defining our variables.

$$\begin{aligned}
\mathbf{r}_1(t) = &< x_1(t), y_1(t) > \rightarrow \text{Parametric Equation to outer boat.}\\
x_1(t) = &\ 10\cos(\pi t) \rightarrow \text{X-component equation to outer boat.}\\
y_1(t) = &\ 10\sin(\pi t) \rightarrow \text{Y-component equation to outer boat.}\\
\mathbf{r}_2(t) = &< x_2(t), y_2(t) > \rightarrow \text{Parametric Equation to inner boat.}\\
x_2(t) &\rightarrow \text{X-component equation to inner boat.}\\
y_2(t) &\rightarrow \text{Y-component equation to inner boat.}
\end{aligned}$$

We first note that the speed of the outer boat (B1) is 10π. Recall that speed is the magnitude of velocity. Taking the magnitude of the derivitive of $\mathbf{r}_1(t)$ confirms this observation. So we know the speed of the inner boat (B2) is 5π. Since the only known values are the boats' initial positions, speeds, and the position equation for B1, we look for a relationship that will allow us do develop B2's position equation. As stated, velocity is the derivitive of position, likewise position is the integral of velocity. We develop a system of differential equations to represent the velocity of B2 in reference to the current position of B1 and B2. We note that since B2 always travels in the direction of B1, the direction of B2 is the difference of B1 and B2's current position.

$$x_2'(t) = x_2(t) - x_1(t) \tag{1}$$
$$y_2'(t) = y_2(t) - y_1(t) \tag{2}$$

These equations give the correct direction, but do not have magnitude of 5π. We must divide by the magnitude of the vector and multiply by 5π. We find the magnitude by

$$\begin{aligned} |\mathbf{r}_2'(t)| &= \sqrt{x_2'(t)^2 - y_2'(t)^2} \\ &= \sqrt{(x_2(t) - x_1(t))^2 - (y_2(t) - y_1(t))^2} \end{aligned} \tag{3}$$

Combining equations 1 and 2 with equation 3 gives the resulting system of equations:

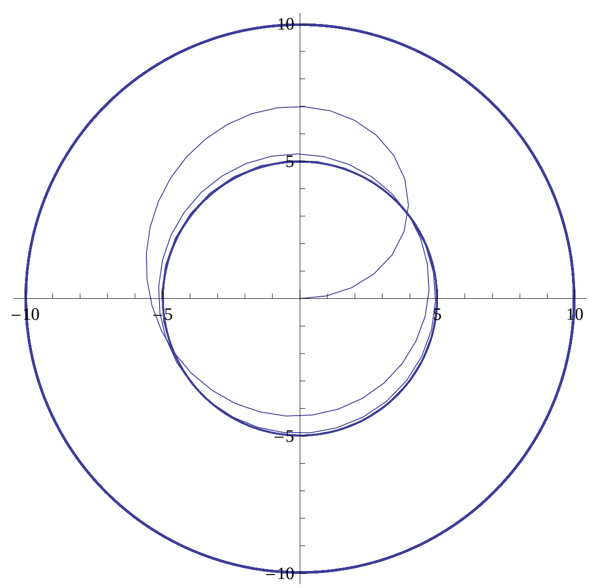
$$x_2'(t) = 5\pi \frac{x_2(t) - x_1(t)}{(x_2(t) - x_1(t))^2 - (y_2(t) - y_1(t))^2} \tag{4}$$

$$y_2'(t) = 5\pi \frac{y_2(t) - y_1(t)}{(x_2(t) - x_1(t))^2 - (y_2(t) - y_1(t))^2}. \tag{5}$$

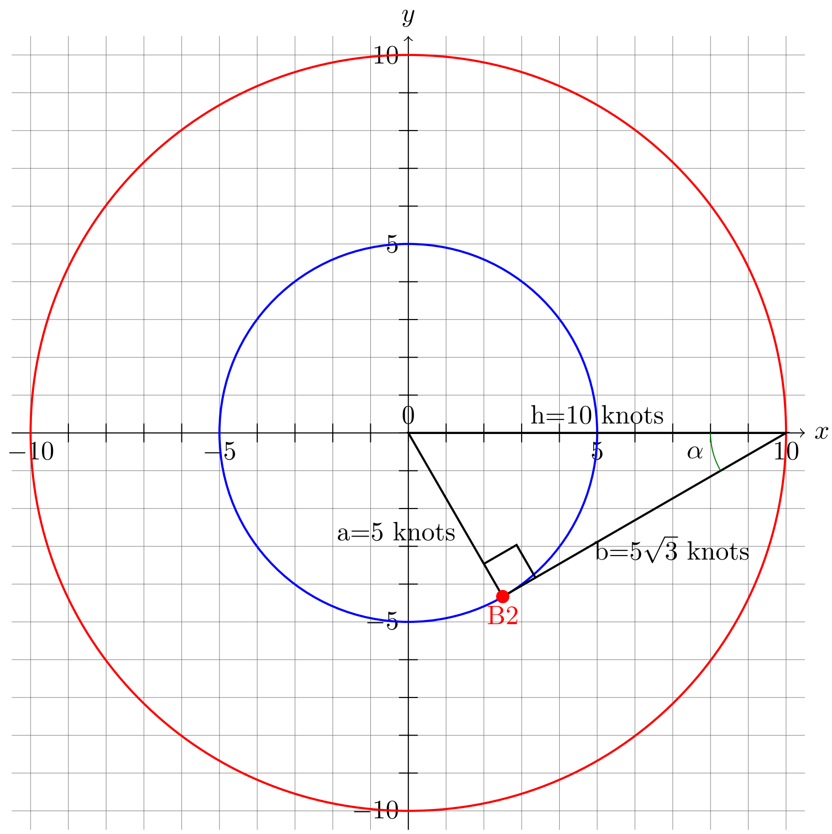
Since we have a non-linear system of differential equations, we use numerical methods (Mathematica) to solve the interpolating polynomial. We can check that the distance travelled by B1 is exactly twice that travelled by B2 using the formula for arc-length.

$$\begin{aligned} L &= \int_0^{200} \sqrt{(x_1'(t))^2 + (y_1'(t))^2} dt \\ &= 2 \int_0^{200} \sqrt{(x_2'(t))^2 + (y_2'(t))^2} dt \\ &= 3141.59 \end{aligned}$$

So evaluating the solution at 200 hours, we find the position of B2 is (2.5,-4.333). Looking at a plot of the position of the two boats (below) show that B2 enters into a circular pattern with radius exactly half that of B1.



Had we plotted the position first, or by some other way realized that B2 would pick up a circular pattern with radius 5, we could have solved the problem geometrically. The direction of B2 is the slope of the tangent line to the circle $x^2 + y^2 = 25$. Since B1 is always at the coordinate (10,0) for every even integer value of t, we construct the following picture.



Now we need to find the x and y projections of a, to determine the coordinates of B2. So the x and y coordinates of B2 are obtained by the following equations.

$$\begin{aligned} x_2 &= a\cos(\alpha) && (6) \\ &= 5\cos(-60) && (7) \\ &= 5\frac{1}{2} && (8) \\ &= \frac{5}{2} && (9) \end{aligned}$$

and

$$\begin{aligned} y_2 &= a\sin(\alpha) && (10) \\ &= 5\sin(-60) && (11) \\ &= 5\frac{-\sqrt{3}}{2} && (12) \\ &= -\frac{5\sqrt{3}}{2} && (13) \end{aligned}$$

So the coordinates for B2 are $(\frac{5}{2}, -\frac{5\sqrt{3}}{2})$.