Problem 16 : Triangle in a Parabola

USMA D/Math Problem of the Week - AY2008

Submission Deadline: March 13, 2008 at 1600

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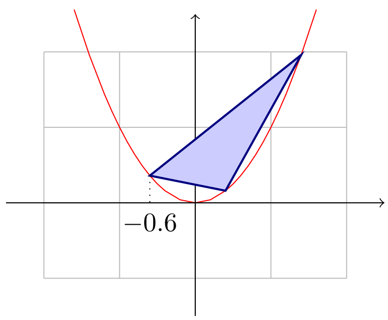
Problem Statement:

Show that any triangle inscribed in the parabola $y = x^2$ whose vertices have x-coordinates $\{a, a+1, a+2\}$ has unit area. One example is shown below (with $a = -0.6$):



Solution:

We begin by defining our variables.

$$
\begin{aligned}
P_1 &= (a, a^2) && \text{Point 1}\\
P_2 &= (a+1, (a+1)^2) && \text{Point 2}\\
P_3 &= (a+2, (a+2)^2) && \text{Point 3}\\
b &= |\overrightarrow{P_1P_2}| && \text{Base}\\
h &= |\overrightarrow{P_3P_4}| && \text{Height}\\
A &= \frac{1}{2}bh && \text{Area}
\end{aligned}
$$

Finding the length of the base is the easy step, so we start there. Using the distance equation, we find the length of the base to be

$$
\begin{aligned}
b &= \sqrt{(a-(a+1))^2 + (a^2-(a+1)^2)^2}\\
&= \sqrt{2}\sqrt{2a^2+2a+1}. \qquad (1)
\end{aligned}
$$

Now the hard part is finding the length of the base. We need the distance formula, but we are only given one endpoint, P_3. We must find the fourth point P_4, such that the vector $\overrightarrow{P_3P_4}$ is perpendicular to $\overrightarrow{P_1P_2}$. Lets begin by finding the equation of the line that passes through P_1P_2 using the point-slope equation of a line.

$$
\begin{aligned}
y &= m(x-x_0) + y_0 \qquad (2)\\
&= \frac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0)+y_0\\
L_1 &= \frac{(a+1)^2-(a)^2}{a+1-a}(x-a)+a^2\\
&= (2a+1)x - a^2 - a \qquad (3)
\end{aligned}
$$

Again, we use equation 2 to find the perpendicular line that passes through P_3 by taking the negative reciporacol slope of line L_1.

$$\begin{aligned} L_2 &= \left(\frac{1}{2a+1}\right)(x-(a+2))+(a+2)^2 \\ &= \left(\frac{1}{2a+1}\right)x+\frac{2a^3+9a^2+13a+6}{2a+1} \end{aligned} \tag{4}$$

Setting equations 3 and 4 equal to each other, we solve for the x component of P_4; the intersection of the two perpendicular lines.

$$\begin{aligned} (2a+1)x-a^2-a &= \frac{1}{2a+1}x+\frac{2a^3+9a^2+13a+6}{2a+1} \\ x_4 &= \frac{2a^3+6a^2+7a+3}{2a^2+2a+1} \end{aligned} \tag{5}$$

Evaluating L_1 or L_2 at the point x_4 yields the y coordinate to P_4.

$$y_4 = \frac{(a+1)(2a^2+6a+3)}{2a^2+2a+1} \tag{6}$$

We are now ready to use the distance equation (and an insane amount of tedious algebra/arithmetic; or Mathematica) between the points P_3 and P_4 to find the length of the height.

$$\begin{aligned} h &= \sqrt{(x_3-x_4)^2+(y_3-y_4)^2} \\ &= \sqrt{2}\sqrt{\frac{1}{2a^2+2a+1}} \end{aligned} \tag{7}$$

Now using the equation for the area of a triangle, we find the area of any "Triangle in a Parabola" does in fact have area equal to one.

$$\begin{aligned} A &= \frac{1}{2}\left(\sqrt{2}\sqrt{2a^2+2a+1}\right)\left(\sqrt{2}\sqrt{\frac{1}{2a^2+2a+1}}\right) \\ &= \frac{\sqrt{2}\sqrt{2}}{2}\sqrt{\frac{2a^2+2a+1}{2a^2+2a+1}} \\ &= \frac{2}{2} = 1 \end{aligned}$$