Problem 6: Sets and Subsets

USMA D/Math Problem of the Week

Submission Deadline: 23 October, 2008 at 1600

Circle one:	cadet	faculty	other

Problem Statement:

Let S be a subset of the set $\{1, 2, 3, \ldots, 2008\}$ such that no two members of S differ by 6 or 11. What is the largest number of elements S can have?

Solution:

To begin, we notice that if we consider numbers modulo 17, then for a pair $\{a, b\}$ if $a - b = 6$ then $b - a = 11$. We use this observation to come up with the following strategy:

1. Maximize the size of such a subset of the set $\{1, 2, 3, \ldots, 17\}$.
2. Repeat this subset. So if $S_1 = \{x_1, x_2, x_3, \ldots, x_k\}$ is the maximum subset we've found, then $S = \{17n + x_i \leq 2008\}$.
3. Since $2008 = 118 \cdot 17 + 2$, we will need to try to include the numbers 2007 and 2008 in the set to get the most possible.

To construct the set S_1 we first need to know how big it can be. Clearly, the set $\{1, 2, 3, 4, 5, 6\}$ satisfies the property that the pairwise differences are neither 6 nor 11, but can $|S_1| > 6$? In fact, the maximum size of S_1 is 8. In order to see this, consider the following pairs of numbers:

$$\begin{array}{ccc} \{1,7\} & \{2,13\} & \{3,9\} \\ \{4,15\} & \{5,11\} & \{6,12\} \\ \{8,14\} & \{10,16\} & \end{array}$$

From the choice of these pairs, it is clear that at most one number from every pair can be in the set S_1, so if $|S_1| = 9$ the only remaining number, 17, must be in S_1. This leads to a chain of implications:

- 11 is not in S_1.

- 5 is in S_1.
- 16 is not in S_1.
- 10 is in S_1.
- 4 is not in S_1.
- 15 is in S_1.
- 9 is not in S_1.
- 3 is in S_1.
- 14 is not in S_1.
- 8 is in S_1.
- 2 is not in S_1.
- 13 is in S_1.
- 7 is not in S_1.
- 1 is in S_1.
- 12 is not in S_1.
- 6 is in S_1.

However, the observation that 6 is in S_1 leads to a contradiction, since the difference between 6 and 17, our starting member of the set, is 11. Thus, $|S_1| < 9$. One set S_1 with 8 members is

$$\{1, 2, 4, 6, 9, 11, 14, 16\}.$$

Other sets would also work. The advantage of this set S_1 is that it includes the numbers 1 and 2, so when we iterate our set, 2007 and 2008 will be included, thus maximizing the number of elements in S. Specifically, S will contain $118 \cdot 8 + 2 = 946$ elements. We note that if there is a set S containing more than 946 elements, then there is a sequence of 17 integers $X = \{x, x + 1, x + 2, x + 3, \ldots, +16\}$ where $|X \cap S| > 8$. However, replacing the numbers 1 through 17 above with X by the mapping $i \mapsto i + x - 1$ leads to the same contradiction, so no set X can have more than 8 members in S. The chosen set has every set X with exactly 8 members, so it must be the best possible.