Problem 7: Football Players

USMA D/Math Problem of the Week

Submission Deadline: 30 October, 2008 at 1600

Circle one:	cadet	faculty	other

Problem Statement:

Twenty-three people of positive integral weights decide to play football. They select one person as referee and then split up into two 11-person teams of equal total weights. It turns out that no matter who the referee this can always be done. Prove that all 23 people have equal weights.

Solution:

(Proof by minimal counterexample)

Assume on the contrary that there is a set of 23 not all equal integer weights satisfying the conditions of the problem. Then among suct sets there is a set $A = (a_1, a_2, \ldots, a_{23})$ with the smallest total wight $w = a_1 + a_2 + \cdots + a_{23}$. If a_i is referee, then $w - a_i = 2s_i$, where s_i is the total weight of each team. Hence $a_i \equiv w(\text{mod}2)$, that is, a_i's have the same parity.

If the a_i's are all evan, we can replace A by

$$A' = \left(\frac{a_1}{2}, \frac{a_2}{2}, \ldots, \frac{a_{23}}{2}\right)$$

a set of less total wight that satisfies the conditions of the problem. And since the a_i's are not all equal, the $a_i/2$'s are not all equal. This contradicts the fact that A is a set with minimum total weight. If the a_i's are all odd, we can use

$$A'' = \left(\frac{a_1+1}{2}, \frac{a_2+1}{2}, \ldots, \frac{a_{23}+1}{2}\right)$$

to lead to a similar contradiction.

Hence our assumption was wrong and all 23 people must have equal weights.

Source (problem and solution): *102 Combinatorial Problems*, Titu Andreescu, Zuming Feng