Chapter 1

Discrete Dynamical Systems

1.1 Introduction

1.1.1 A Mathematical Model

This course and book focus on the use of the power of mathematics to solve important (often urgent), real world problems. The central concept of this book is *mathematical modeling*. A **mathematical model** is a construct (e.g., a function or equation) that is designed to predict the behavior of a system. Because assumptions are often made, a model usually is an idealization of the actual system it represents. Certainly, all models have limitations. In fact, Dr. George Box, one of the most prominent statisticians of the 20th century, is credited with having said "all models are wrong - but some are useful."[1]

Definition 1 *A* **mathematical model** *is a construct (e.g., a function or equation) that is designed to predict the behavior of a system.*

Mathematical models can be used to solve problems in many different kinds of situations: to predict how a population may grow, estimate life expectancy, or determine how much a manufacturer should charge for a product. Other important examples of mathematical models include financial models and weather models. Meteorologists create computerized mathematical models that attempt to predict weather patterns. One pattern that meteorologists are always trying to predict is the path of hurricanes. Figure 1.1 illustrates a projected path of Hurricane Ike in the 2008 hurricane season; it was developed using a mathematical model.

Figure 1.1: Track of Hurricane Ike in 2008

[1]Box, George. E. P. Robustness in the Strategy of Scientific Model Building. In R. L. Launer, and G. N. Wilkinson, (eds.) Robustness in Statistics. New York: Academic Press

Also, in the next few lessons you will develop many different financial models that will enable you to answer questions like how much money you need to save to become a millionaire by the time you retire from the military or how you should invest your cow loan. You will find that becoming a millionaire is quite achievable.

Example 1 *Suppose you are playing a game in which you are guessing the heights of students that walk into the classroom. The good news is that you get to collect data to help you guess. The data you gather is below.*

$CDT1$	$CDT2$	$CDT3$	$CDT4$	$CDT5$
60”	72”	68”	62”	66”

$CDT6$	$CDT7$	$CDT8$	$CDT9$	$CDT10$
78”	70”	72”	67”	61”

Recall from the first page of the text that a model “is a construct (e.g., a function or equation) that is designed to predict the behavior of a system.” Based upon that definition and the data above, how tall would you guess the next cadet entering the room would be? Suppose that you were to earn 10 bonus points for guessing the answer exactly, nine for being off by an inch, 8 for two inches, etc. In this case, picking an extreme height could be disastrous for you - causing you to earn no bonus points. How can you maximize the possible number of bonus points?

The answer lies in your first mathematical model for the year - the *average*, or *mean* of the data. In this case, the model will be too high half the time and too low half the time, but you are as safe as you can be against either an extremely tall or short cadet entering the room.

$$Model = average = \frac{(60 + 72 + 68 + 62 + 66 + 78 + 70 + 72 + 67 + 61)}{10} = 67.6$$

Therefore, the model (and the “safe” prediction) for the height of the next cadet to enter the room is 67.6 *inches*.

Is it possible to make a model that is better than the average? Of course it is! In fact, in future section we will begin learning about how to develop models that can be much more precise than simply taking the average. We will also learn how to compare models to determine which is better.

1.1.2 Problem Solving Processes

As problem solvers, we all have different levels of expertise and experience. Many problems appear complex or unsolvable at first - perhaps some are. We can, however, gain greater insight about a problem. There are many well-known methods or problem solving processes. Perhaps you recall a method that you used in high school, but didn't call it a process. Our minds often disect a problem into manageable steps to help us arrive at a logical conclusion. A problem solving process can help structure our thoughts as we solve a problem. Using a process can help ensure that we do not leave out crucial information and that we consider all alternatives.

A problem solving process often used in the military is known as the Military Decision Making Process (MDMP). The MDMP (Figure 1.2) illustrates the way military officers go from receiving a mission from a higher authority to execution of the mission at their level.

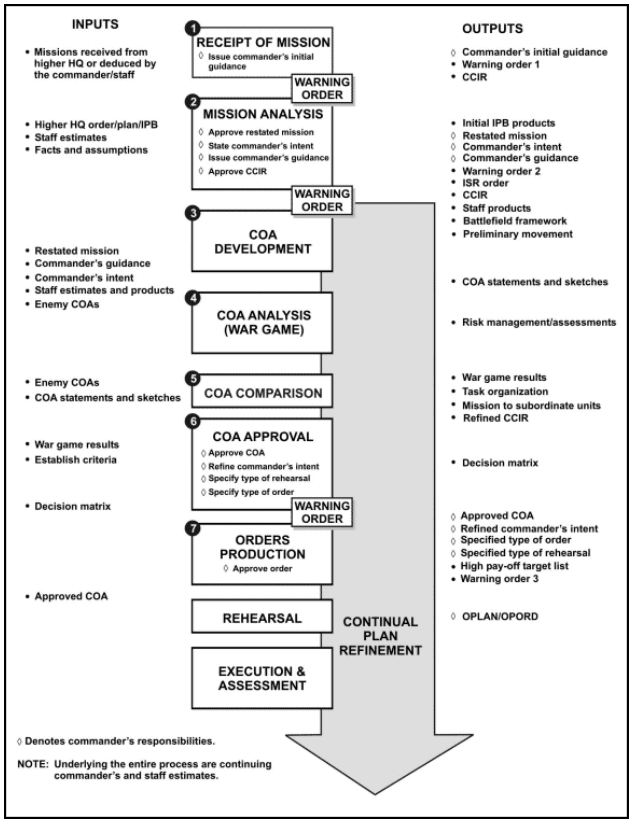

Figure 1.2: Military Decision Making Process

The problem solving process that we highlight combines the process developed by Dr. George Polya in his book, How to Solve it, and the process developed by the USMA Chemistry Department, known as Given, Find, Plan, Solve, and Reflect (GFPS-R). This process is described below.

- **Given.** Understanding what is given in the problem is the first step in solving the problem.
  - What is known?
  - What is unknown?
  - Define the variables of interest.
  - What assumptions must be made about the unknowns to solve the problem? **All** assumptions must be **valid** and **necessary**.
    - Valid: The assumption is accurate (e.g., assuming that a parabola would be an appropriate model to describe the path of a punted football).
    - Necessary: The assumption is required for the problem (e.g., not "the golf ball is orange" if we are determining how far Tiger Woods can drive a ball).
  - Visualize the situation and draw a picture, if possible.
  - Identify the units to be used throughout the problem.
- **Find.** What do we need to *find* to solve the problem?
- **Plan.** Develop an idea of what solution techniques will be most appropriate in the problem and how you will apply them.
  - Have you worked this problem or a similar problem before? Your previous solution process may help this time, too.
  - If necessary, break the problem into smaller parts.
  - Ensure you account for all the important information in the problem - do not disregard important data or an important condition.
  - Identify the fundamental concepts and tools needed to solve the problem.
  - Identify the steps you will follow to solve the problem.
  - Determine the role of technology in your solution process.
  - Write down an estimated answer.
- **Solve.** Carry out the solution plan. *Check each step!*

- **Reflect.**
  - Examine the solution obtained. *Does it make sense?* Is it close to your estimate? If not, why?
  - Ensure the units are correct.
  - Consider the assumptions that you made: given your solution, do they make sense? Do the assumptions need adjusting? How sensitive is your solution to changes in the assumptions (see Sensitivity Analysis on page 25)?

1.1.3 Mathematical Modeling

In Section 1.1.1 we defined what a mathematical model is. Mathematical modeling then is the art of creating mathematical models. It is more than just creating a function or a model. Mathematical modeling incorporates all of the key elements of problems solving and is iterative in nature. Figure 1.3 shows the incorporation of these problem solving elements into the iterative modeling process that we will use throughout this course. The steps described in Figure 1.3 are explained below.

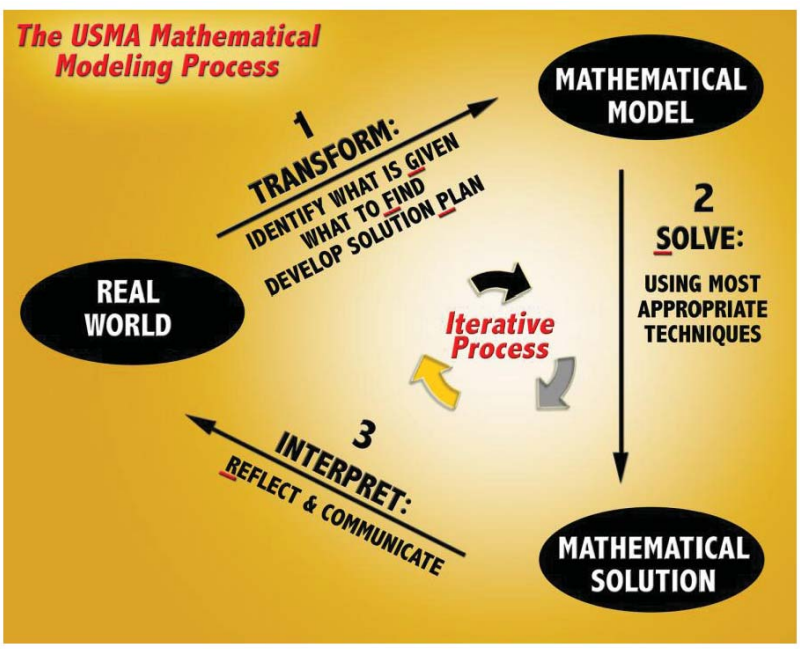

Figure 1.3: The Mathematical Modeling Triangle

- **Step 1**. *Transforming the Real World Problem.* Before developing the model, a thorough understanding of the problem at hand must be gained. Understand the root of the actual problem, not the symptoms of the problem. Know what is *given* and what you need to *find*. When this understanding is achieved, *transform* the problem into a mathematical model that can be solved using quantitative techniques. It is in the transformation of the problem into a mathematical model that you begin to develop the "plan of attack" to solve the problem. A model developed for one situation may not work in another. Beginning to think about the plan during the transformation step may save time later.

- **Step 2**. *Solve.* When the model is complete, finish the solution plan. After formulating the model, apply the most appropriate solution techniques to provide the desired answer. As with model selection, you will see in this course that choice of solution technique can be the difference between attaining an answer to a problem and not finishing.

- **Step 3**. *Interpret the Solution.* After obtaining a solution, a vital step in the modeling process is to reflect on the results. Does the solution make sense and does it solve the problem posed? Does it reveal another problem that must be solved? If another problem is revealed, proceed around the modeling triangle as often as needed to develop a complete solution. Reflect on the impact of any assumptions made. After reflecting on the solution, interpret the answer: determine what the answer means in the context of the problem and translate it into simple, non-mathematical language for the decision-maker.

We can demonstrate this process with a simple problem shown below.

Example 2 *John was born three years before his sister Joan and is now twice as old as she is. How old are John and Joan?*

- **Step 1**. *Transform the Problem.* What is **given**? We see the relationship between the ages of John and Joan. What do we need to find? John's and Joan's ages. Variable definitions: the letters x and y denote the age of John and Joan, respectively. Next, we must *transform* the information that we know into a mathematical model that we can solve quantitatively.

  - John was born three years before his sister, therefore his age is three greater than his sister, $x = y + 3$.
  - John is twice as old as his sister, therefore his age is twice his sister's: $x = 2y$.
  - Our plan of attack looks to be the solution of a system of equations.

- **Step 2**. *Solve using the most appropriate techniques.* We have two equations and two variables (unknowns) for which to **solve**. Our plan is to use substitution, let's implement the plan.

  - If $x = 2y$, we can use that relationship to put the equation $x = y + 3$ in terms of one variable and solve: $2y = y + 3$. Subtracting y from both sides results in $y = 3$
  - Since we know that $x = 2y$, we can calculate x by substituting in $y = 3$: $x = (2)(3) = 6$.

- **Step 3**. *Interpret the solution.* It is now time to *interpret* the results in a manner that it easily understood. Since x represented John's age and y represented Joan's age, we answer the original question posed by stating "John is 6 years old; Joan is 3 years old." *Reflect*: our solution satisfies all of the conditions of the problem.

The following set of word problems is intended to provide practice using the modeling triangle to solve familiar kinds of problems. Our goal in this course is to use teh modeling triangle for solving more complicated problems than those posed here. However, it is useful to begin with less complex problems and then increase the level difficulty of the problems.

Question 1 *You are working with an aerial photograph taken by an unmanned aerial vehicle (UAV) of a meeting between an unknown enemy operative and a known enemy operative. The meeting was outside in bright sunlight. The known operative's height is 5 feet, 8 inches and the length of his shadow on the ground is three feet. The length of the unknown operative's shadow is 3 feet, 6 inches. You would like to know his height to help identify him at a later time. Solve this problem using the three steps of the mathematical modeling triangle.*

- *Understand the problem and transform it into a mathematical model*
- *Use appropriate solution techniques to attain a solution to the problem*
- *Reflect on your solution, then communicate it in easily understandable terms*

For this problem, it is helpful to draw a sketch showing the situation. Note the two triangles in Figure 1.4 on page 9 are similar because the shadows caused by the light rays from the sun are parallel. The lengths in this figure are expressed in inches.

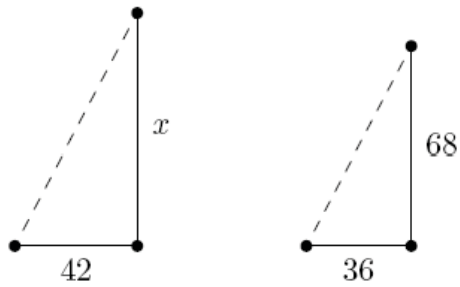

Figure 1.4: Relationship between Height and Shadow Length

Question 2 *You are building a recreation center whose water supply will come from a well that is 150 feet east and 200 feet north of the recreation center. You will need to order pipe to connect the well to the recreation center. The pipe comes in 25 foot sections. How many sections do you need to order? Assume that there are no obstacles to burying the pipes and that they may be laid in a straight line from the well to the recreation center.*

Question 3 *Suppose that you want to make your squad leader's birthday special. Your squad mates and you have decided that to make it REALLY special, you will need to fill his room with balloons. We aren't just talking about a couple of balloons that say Happy Birthday in bright pink letters - you want the balloons to fill all available space in his room. Would a couple of hundred balloons do it? Maybe a thousand balloons? How long will it take to do this and how much will it cost?*

Question 4 *How could you make the estimate of the number of balloons required in Question 3 better? Describe the method and determine the total number of balloons required using your improved method.*

Question 5 *How many doctors are there in the city of Boston?*

Notice these problems, like most of the problems you will encounter in this book, are located throughout the section, rather than at the end of the section. Learning mathematical modeling is not a spectator sport. You will get the most out of this text by doing the problems as they are posed in the text. These problems are strategically placed to reinforce the concepts that immediately precede them. You will read about a concept to gain familiarity, and then do the problems to reinforce your conceptual understanding before moving on to the next concept.

1.1.4 Why Mathematical Modeling is Important

A goal of this book and the United States Military Academy core mathematics program is to leverage the power of mathematics to help gain insight into the solutions of important and often urgent problems. In this course, previous students have developed mathematical models that have solved the following problems:

- Will a bridge collapse when a truck with a certain weight drives over it?
- What is the investment portfolio that will provide the most money in retirement, considering risk tolerance?
- How soon will a lake poisoned by contaminant be clean enough to provide a village with water?
- Given a diagram of an oceanic shipping port, where should I put the most protection to keep the port safest against attack?
- How does mathematics create image transformations seen in animated movies?
- How much money can I expect to earn if I play roulette at a casino?
- What is a viable schedule for a diabetic to follow to guide eating and insulin usage to maintain blood sugar in a healthy range?

Question 6 *How much peanut butter does the Corps of Cadets eat in a year?*

Question 7 *What aspects do the problem solving processes discussed in this text (as well as any others that you are aware of) have in common?*

Question 8 *Compare and contrast the modeling process that we have explained here with the process described by Stewart in your Calculus text on page 24.*