## 1.2 Introduction to Discrete Dynamical Systems

This chapter is about modeling with "discrete dynamical systems." The word "discrete" refers to objects that "consist of distinct or unconnected elements,"[2] and the word "dynamic" refers to situations where the quantities of interest are changing over time. By the end of this section, you should be able to answer these questions.

- What are "discrete" dynamical systems and how is this kind of modeling different from other kinds of modeling?
- When should I use "discrete" dynamical systems and when should I use other kinds of modeling?

In this section we contrast "discrete" dynamical systems and "continuous" dynamical systems. We begin this discussion with two examples. The first example is best modeled using "discrete" dynamical systems and the second is best modeled using "continuous" dynamical systems. Looking at these two models side-by-side helps answer the two questions above.

**Example 1** *A country, named Tankerland, is building up supplies of oil for a possible military engagement. Every month a tanker arrives with 500,000 tonnes of oil that are added to the reserves. The current reserve has 1,500,000 tonnes and the first tanker is scheduled to arrive at the end of the first month. The graph on the left side of Figure 1.5 shows the oil reserves in Tankerland over the next year.*

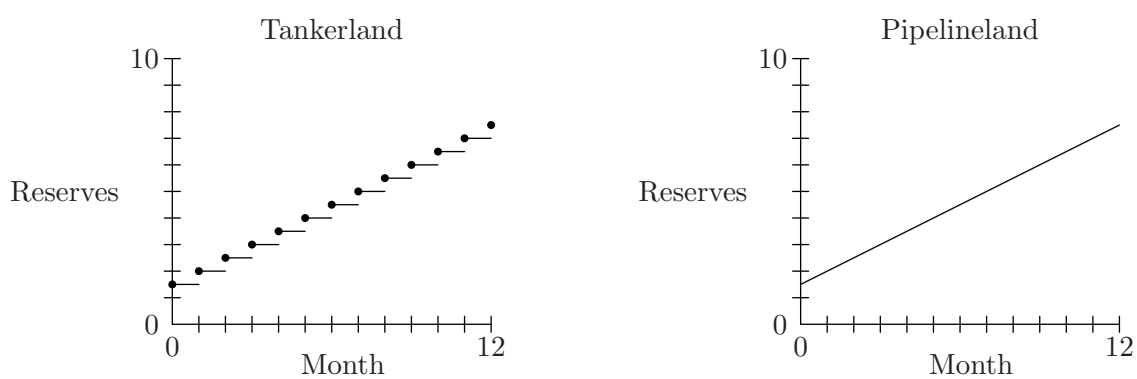


Figure 1.5: Tankers and a pipeline (reserves in millons of tonnes)

[2]Webster's Ninth New Collegiate Dictionary, Merriam-Webster, Inc, 1983

**Example 2** *A neighboring country, named Pipelineland, is building up supplies of oil for a possible military engagement. This country has its own oil wells and is able to deliver oil via a pipeline directly to its reserves at the rate of 500,000 tonnes per month. The current reserve has 1,500,000 tonnes. The graph on the right side of Figure 1.5 shows the size of the reserves in Pipelineland for the next year.*

Even though 500,000 tonnes of oil are being added to the reserves each month in both Tankerland and Pipelineland, the two graphs look different. The tankers deliver oil at **discrete** times in big "chunks" and the pipeline delivers oil in a **continuous** stream.

For Example 1 we use a **discrete** dynamical system to model the amount of oil entering Tankerland. We will use the letter $n$ to denote time. The values of $n$ will be $0, 1, 2, \ldots 12$. We use the notation $p_n$ to denote the amount of oil in the reserves during month $n$ up to but not including the very last day. Because oil is delivered on the very last day of each month, the amount of oil in the reserves jumps on the very last day of each month. This gives us the model

$$\begin{aligned} p_0 &= 1.5 \text{ million tonnes} \\ p_1 &= p_0 + 0.5 = 2.0 \text{ million tonnes} \\ p_2 &= p_1 + 0.5 = 2.5 \text{ million tonnes} \\ &\vdots \\ p_{12} &= p_{11} + 0.5 \text{ million tonnes} \end{aligned}$$

The ordered collection of numbers, $p_0, p_1, \ldots p_{12}$ is called a **sequence**; specifically, an **arithmetic** sequence. We will explore the behavior of both **arithmetic** and **geometric** sequences (and those that are neither) in future sections. The individual numbers in a sequence are called the **terms** of the sequence. Notice the pattern in the calculations above – starting with the first term (or **initial value**), $p_0 = 1.5$, we compute each subsequent term by

$$p_n = p_{n-1} + 0.5, \quad \text{for } n = 1,\ 2,\ \ldots,\ 12$$

This equation, or pattern, is called a **recursion equation**. One way to read this particular recursion equation is "each month we add 0.5 million tonnes of oil to the previous amount in the reserves." Notice the parallel between the mathematical statement "$p_n = p_{n-1} + 0.5$" and the English statement "each month a tanker delivers 0.5 metric tonnes of oil."

**Definition 1** *A* **recursion equation**, $p_n$, *is an equation that describes how the terms of a sequence are computed.*

Normally, the current term, $p_n$ will be a function of the previous term, $p_{n-1}$. For example,

$$p_n = p_{n-1} + 5$$

is a **recursion equation** that computes the current term, $p_n$, using the term that immediately precedes it, $p_{n-1}$. There are other examples such as

$$p_n = p_{n-1} + 5p_{n-2} - 1.2$$

that compute the current term, $p_n$, using the more than one preceding term.

For the second example we use the **continuous** model

$$p(t) \quad = \quad 1.5 + 0.5t \text{ million tonnes}, \quad \text{for } t \in [0, 12]$$

The function $p(t)$ represents the amount of oil in the reserves $t$ months from now.

We use the words **discrete** and **continuous** to distinguish between models in which the quantity of interest changes at distinct times versus models in which it changes at all instants of time. Because the idea of change is so important, we often write a model like this in the form of a **difference equation**

$$p_n - p_{n-1} \quad = \quad 0.5 \text{ million tonnes} = \frac{\Delta \ oil \ reserve}{month}$$

to emphasize the change from one month to the next. For a continuous model the same idea is written as a **differential equation**

$$\frac{dp}{dt} \quad = \quad 0.5 \text{ million tonnes} = \frac{\Delta \ oil \ reserve}{month}$$

The symbol $\frac{dp}{dt}$ on the left side of this equation denotes the **derivative**. The derivative is the instantaneous rate of change of a continuously changing quantity. For example, if $p(t)$

is the location of a car at time $t$, then its derivative $\frac{dp}{dt}$, represents the car's velocity. We study derivatives and continuous models later in this course.

Notice the parallel between the mathematical statement "$\frac{dp}{dt} = 0.5$" and the English statement "oil is flowing into the reserves at the rate of 0.5 metric tonnes per month."

In summary, we will eventually use two different kinds of dynamical systems:

- **Discrete dynamical systems**
  - The quantity of interest changes at *distinct* times and is described by a sequence – $p_0, p_1, \ldots$ (arithmetic, geometric, or neither).
  - A graph of the quantity of interest looks like the graph for Tankerland in Figure 1.5 on page 11 and jumps at distinct times.
  - The way in which the quantity of interest *changes* is described by a **difference equation** – for example,

    $$p_n - p_{n-1} = 0.5$$

    or by a **recursion equation** – for example,

    $$p_n = p_{n-1} + 0.5.$$

    Notice that the recursion equation and the difference equation express exactly the same information. The difference equation is often read "present - past = change" and focuses our attention on the difference between the present and the past. This particular recursion equation might be read "present = past + change" and focuses our attention on how we determine the present based on the past.
  - Each model contains an **initial value** that tells us the starting value of the quantity of interest.
- **Continuous dynamical systems**
  - The quantity of interest, described by a function – $p(t)$, changes at *all* instances of time.
  - A graph of the quantity of interest looks like the graph for Pipelineland in Figure 1.5 on page 11 and shows continuous change with no jumps.

- The way in which the quantity of interest changes is described by a **differential equation** – for example,

$$\frac{dp}{dt} = 0.5.$$

- Each model contains an **initial value** that tells us the starting value of the quantity of interest.

It is worthwhile to look at the "modeling triangle" again, shown in Figure 1.6.

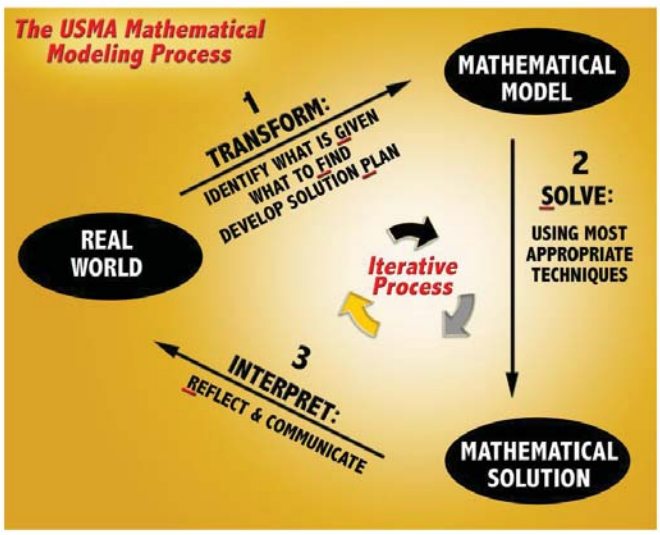


Figure 1.6: The Mathematical Modeling Triangle

The heart of this book is using mathematics as a language. We transform problems from the real world into mathematical problems, solve those problems mathematically, and then interpret the mathematical results back in the real world. Because we are interested in using mathematics to solve real and important problems, the arrows from and to the "*real world*" circle are as important as the mathematical manipulations represented by the arrow on the right from the "*mathematical model*" to the "*mathematical solution*." In fact, these parts, transforming real problems into mathematical expressions and interpreting mathematical results back in the real world, of modeling and problem-solving often require a better understanding of mathematics than the purely mathematical solutions.

We often use a formula to describe a model – for example, in Tankerland the amount of oil in the reserves after the $n$-th delivery is given by the formula $p_n = 1.5 + 0.5n$ and

in Pipelineland the amount of oil in the reserves is given by the formula $p(t) = 1.5 + 0.5t$. Except for using the notation $n$ in one formula and the notation $t$ in the other, these two formulas look identical. But, we must be careful and remember the underlying model structures: the Tankerland model is discrete and the Pipelineland model is continuous. Therefore, the values for the domain of $n$ will be discrete, and the values for $t$, will be continuous.

We close this section by listing explicitly five steps that are very useful for developing a model. These steps are all part of the *transform* arrow in the modeling triangle (Figure 1.6 on page 15) – expressing a situation in the real world mathematically. Following these steps carefully can help you organize your model development and avoid some common mistakes. Notice that we list these steps using a bulleted list rather than a numbered list because you may not always follow these five steps exactly in the order below.

- **Define the variables of interest**. These variables identify the quantities of interest – that is, the quantities that you are modeling or tracking. For example, you might be interested in the amount of money in a bank account, the average personal income of the residents of a particular country, or the amount of oil in worldwide known oil reserves. As part of identifying and defining the variables of interest, you need to describe the units in which they are measured. For example, the balance in a retirement account that accrues interest monthly can be annotated $p_n$ where $n$ is measured in months after the initial investment.

- **Identify the domain**. In most of the models in this book, we are tracking how the variables of interest change over time. This is not, however, always the case. For example, we might be interested in tracking how the amount of dissolved oxygen in the ocean changes as the depth changes. That is, we use the notation $p_n$ for the value of $p$ at time $n$ or at depth $n$. We need to specify the units in which $n$ is measured and the domain of $n$. For example, if we are tracking the amount of money owed on a car loan over a period of 60 months, we would use the notation $p_n$ to denote the amount of money owed after $n$ monthly payments. The subscript $n$ would be measured in payments or months and $n$ would have the values $n = 0, 1, 2, 3, \ldots 60$. The number $p_0$ would represent the initial amount borrowed ... and the number $p_{60} = \$0.00$ would represent the final loan balance after the 60th, and final, loan payment.

- **List assumptions**. It is easy to go overboard listing assumptions but it is also easy to err in the other direction, ignoring assumptions that are important. For example, if we are modeling the price of gasoline we might assume that political conditions in oil-producing countries are stable. Remeber that assumptions must be valid and necessary.

- **State the initial value**. If, for example, a model tracks the amount owed on a $\$15,000$ loan, the initial value would be $p_0 = \$15,000$.

- **Determine the difference equation or the recursion equation**. If, for example, the loan was paid off in *monthly* installments of \$350 at an *annual* interest rate of 6%, the recursion equation would be

$$\begin{aligned} p_n &= p_{n-1} + \left(\frac{0.06}{12}\right) p_{n-1} - 350 \\ &= 1.005 p_{n-1} - 350, \end{aligned}$$

  and the differnce equation would be

$$p_n - p_{n-1} = 0.005 p_{n-1} - 350.$$

**Question 1** *The M1091 fuel/water tanker (Figure 1.7.[3]) can transport 1500 gallons of either water or fuel. A supply station has been established in a secure area to supply water to troops on the front lines. The stattion has 95,000 gallons of water in a holding tank and every day ten M1091 tankers leave the station with a full load of water. Develop a model for the water in the holding tank. Draw a graph for this model. When will the station need to be resupplied?*

Figure 1.7: M1091 Fuel/Water Tanker

[3]Source: www.globalsecurity.org.

**Question 2** *Water can be pumped from the M1091 at a rate of 100 gallons per minute. Develop a model showing how much water is in an M1091 starting at the moment it starts pumping water out. Draw a graph for this model. How long will it take to completely empty each tanker?*

**Question 3** *You are posted at a mobile command center. Every morning you receive an update from a unit in the field reporting how much fuel they have. This morning they reported that they have 200,000 gallons of fuel. You know that every day they have reported having 15,000 gallons less fuel than the day before. You post the most recent report every morning immediately after it comes in at 0700. Develop a model for the number posted. Draw a graph for this model.*