BLOCK I - ENDSTATE PROBLEMS

By the end of Block I, you should be comfortable solving problems similar to the ones suggested in this section. The problems are labeled as Level 1, Level 2, and Level 3 type problems. The level of difficulty for WPR questions are most similar to the level of difficulty associated with the Level 2 type problems below. Level 1 type problems are most similar to some easier text "Do Problems", and level 3 type problems are comprised of greater difficultly problem solving scenarios. The WPRs ***do*** require you to perform problem solving skills and the questions below are not inclusive of all the Block Objectives.

Level 1

1. The profits of a small company for each year of the first five years of its operations are given in the following table:

Year	Profit (in thousands of dollars)
2000	6
2001	27
2002	62
2003	111
2004	174

   a. What is the average rate of increase of the profits over the first five years of operation? Between 2002 and 2004?

   b. Estimate the rate at which the profits were changing in 2002.

2. $\frac{dy}{dx} = ?$ if $x^2 + y^2 = 25$ (Hint: Use implicit differentiation)

3. Find two numbers whose difference is 1000 and whose product is a minimum.

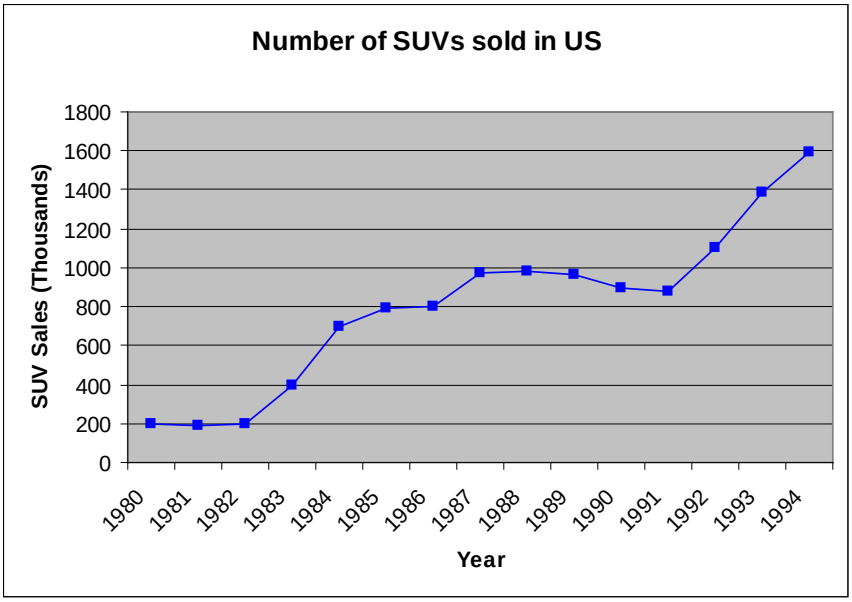
4. A 17-ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 5 ft/sec, how fast will the top of the ladder be moving down the wall when it is 8 ft above the ground?

Level 2

5. Evaluate the following derivatives:

$$f(x) = -2Cos(3x)$$
$$f'(x) = ?$$
$$g(t) = \ln(x^2)$$
$$g'(t) = ?$$
$$r(s) = \sin(6s)/(3s^2 + 4)$$
$$r'(s) = ?$$

6. The table and graph below describe the average sales in the US of SUVs from 1980 to 1994.



Year	SUV Sales (thousands)
1980	200
1981	192
1982	200
1983	400
1984	700
1985	793
1986	800
1987	970
1988	980
1989	963
1990	900
1991	880
1992	1100
1993	1390
1994	1597

a. Determine the average rate of change in sales for SUVs over the given 14 year interval (remember to include units).

b. Estimate the instantaneous sales rate in 1992. Justify your answer and include units.

7. $\frac{dy}{dx} = ?$ if $x^3 + y^3 = 6xy$ (Hint: Use implicit differentiation)

8. Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture such that its volume increases at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.

9. An 1125 cubic foot open-top rectangular tank with a square base x feet on a side and y feet deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made, but also an excavation charge. If the material costs five dollars per square foot, and the excavation charge is $10xy$, find the dimensions of the tank which minimize the total cost.

10. The number of portable CD players you are prepared to supply to a retail outlet every week is given by the formula $q = 0.1p^2 + 3p$, where p is the price it offers you. The retail outlet is currently offering you \$40 per CD player. If the price it offers decreases at a rate of \$2 per week, how will this affect the number you supply[1]?

[1] Stefan Waner and Steven R. Costenoble, *Applied Calculus 3rd edition*. Thomson Brooks/Cole 2004. pp 334.

Level 3

11. Evaluate the derivatives of the following functions:

    a. $y = 5x^2 e^{4x+1} + \ln(\sin(2x^3 - x^2 + 5x - 10))$

    b. $f(x) = e^{\sin(\sqrt{x})}\cos(\pi x) + \sin\left(\dfrac{x^2 - 4}{3x+1}\right)$

12. Concert tickets went on sale at noon (t=0) and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time *t* is modeled by a twice-differentiable function *L* for $0 \le t \le 9$. Values of *L(t)* at various time *t* are shown in the table below.[2]

t (hours)	0	1	3	4	7	8	9
L(t) (people)	120	156	176	126	150	80	0

    a. Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:50 P.M. (*t*=5.5). Show the computation that lead to your answer. Indicate units of measure. Provide a graphical interpretation of your results.

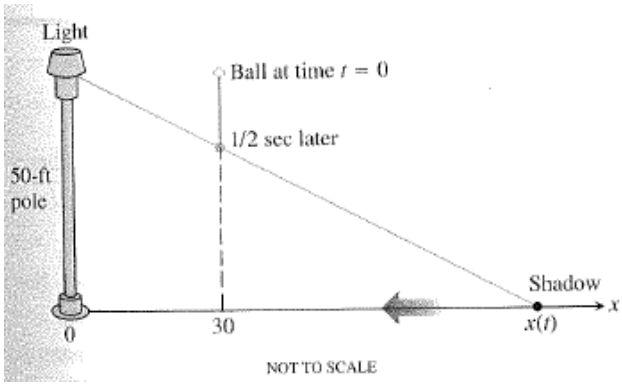
    b. For $0 \le t \le 9$, what is the fewest number of times at which *L'(t)* must equal 0? Give a reason for your answer.

13. What value of *a* makes $f(x) = x^2 + \dfrac{a}{x}$ have:

    a. A local minimum at $x = 2$?

    b. A point of inflection at $x = 1$?

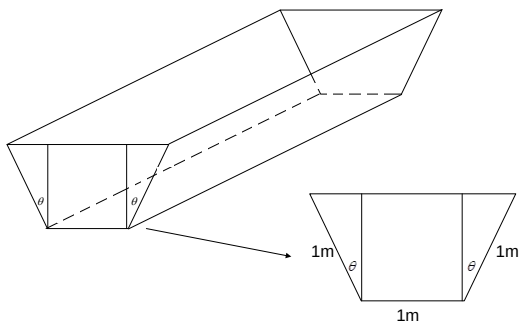
14. A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. How fast is the shadow of the ball moving along the ground 0.5 seconds later? (Assume that the ball falls a distance $s(t) = 16t^2$ ft in *t* seconds).



[2] AP Calculus AB Free-Response Questions, The College Board, 2008.

The *thin lens equation* in physics is: $\frac{1}{s}+\frac{1}{S}=\frac{1}{f}$, where s is the object distance from the lens, S is the image distance from the lens, and f is the focal length of the lens. Suppose that a certain lens has a focal length of 6 centimeters and that an object is moving towards the lens at the rate of 2 centimeters per second. How fast is the image distance changing at the instant when the object is 10 centimeters from the lens? Is the image moving away from the lens or toward the lens?

1. The trough shown below is to be made with the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?



Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

BLOCK I: LIMITS, CONTINUITY, AND SINGLE VARIABLE DIFFERENTIATION

WEEK	MONDAY	TUESDAY	WEDNESDAY	THURSDAY	FRIDAY
1	5-Jan-09	6-Jan-09	7-Jan-09	8-Jan-09	9-Jan-09
	Re-Organization Week			Lesson 1: Intro to Differential Calculus	Lesson 2: Limit of a Function (Review)
2	12-Jan-09	13-Jan-09	14-Jan-09	15-Jan-09	16-Jan-09
	Lesson 3: Continuity (Review)	Lesson 4: Derivatives and Rates of Change (Review)	Lesson 5: The Derivative as a Function	Lesson 6: Derivatives of Power, Exponential, and Log Functions	Lesson 7: The Product and Quotient Rules
3	19-Jan-09	20-Jan-09	21-Jan-09	22-Jan-09	23-Jan-09
	Lesson 8: Derivatives of Trig Functions	Lesson 9: The Chain Rule **(Issue CW Derivative HW)**	**No Class**	**Lesson 10: Derivative WRIT (In Class - 100Pts)**	Lesson 11: Maximum and Minimum Values of Functions
4	26-Jan-09	27-Jan-09	28-Jan-09	29-Jan-09	30-Jan-09
	Lesson 12: Derivatives and the Shapes of Curves	Lesson 13: Solving Optimization Problems 1	**No Class**	Lesson 14: Solving Optimization Problems 2	Lesson 15: Solving Optimization Problems 3
5	2-Feb-09	3-Feb-09	4-Feb-09	5-Feb-09	6-Feb-09
	Lesson 16: PSL 1	Lesson 17: Implicit Differentiation	**No Class**	Lesson 18: Problem Solving with Related Rates 1 **(Issue Part 1 of Project)**	Lesson 19: Problem Solving with Related Rates 2
6	9-Feb-09	10-Feb-09	11-Feb-09	12-Feb-09	13-Feb-09
	Lesson 20: Mathematical Dimensions	Lesson 21: PSL 2	Lesson 22: Project Research Day (In-Class)	**Lesson 23: WPR 1 (Dean's Hour)**	**No Class**

Block Objective:

Understand the derivative as an instantaneous rate of change and know the graphical, algebraic, and physical interpretation of the derivative. Be able to apply the concept of the derivative as an instantaneous rate of change in order to model and solve problems.

Supporting Objectives:

2. Understand the relationship and difference between *average rate of change* and *instantaneous rate of change.* Understand both graphically, numerically, and algebraically.
3. Determine and evaluate the derivative of functions using the properties of the derivative and basic rules (power, trigonometric, etc.). Be able to apply them while using the product, quotient, and chain rule.
4. Find the derivatives of implicitly defined functions.
5. Model and solve problems using related rates of change.
6. Determine the local minima/maxima of a function.
7. Determine the absolute minimum/maximum of a function using the Closed Interval Method.
8. Model and solve single variable optimization problems.

Technology:

1. Define a function in *Mathematica*.
2. Plot a function or several functions over a specified domain in *Mathematica*.
3. Solve a single equation for an unknown variable using *Mathematica*.

4. Evaluate derivatives using *Mathematica*.

Useful *Mathematica* Commands for Block 1:

1. **`?`** or **`??`**
2. **`Plot`**
    a. **`AxesLabel`**
    b. **`PlotRange`**
3. **`ListPlot`**
4. **`Solve`**
5. **`//N`**
6. **`D`**
7. **`ContourPlot`**

Think about: -How will your knowledge of Trigonometry and Geometry help you in this block?
-Why would you want to set the derivative of a function equal to zero?
-What does the notation $y(x) = x$ mean?

Links to Other Pages: Block I: Overview Block I: Endstate Problems

LESSON 1 – INTRODUCTION TO DIFFERENTIAL CALCULUS

OBJECTIVES:

1. Approximate the slope of the tangent line at a point on a given curve using the slopes of secant lines.
2. Approximate instantaneous rates of change using average rates of change.

READ:

1. Instructional Memorandum #104-2 to include enclosures.
2. Stewart: Section 2.1, pages 83-86.
3. Student Notes.

THINK ABOUT:

1. What is the relationship between tangent lines and instantaneous change?
2. Secant lines and average change?

DO: Section 2.1/ 1, 2, 5, 7

STUDENT NOTES:

Welcome to MA104, Differential Calculus. As the name of the course implies, we will study the concept of the derivative, in single and multivariate setting, and then apply our knowledge of the derivative and its relatives, partial and directional derivatives, in order to model and solve problems.

This course is a continuation of what you studied in MA103, and should not be studied without keeping this in mind. MA104 simply extends your knowledge from the world of discrete change into one of continuous change, from average rate of change to instantaneous rate of change.

The fundamental question that lies before us is how to describe the relationship between average rate of change and instantaneous rate of change. At this point in your mathematical career you should be familiar with the concept of average rate of change. We can illustrate this idea with a simple example and an often used rate of change, velocity. If you drive from West Point to New York City, which is approximately 50 miles away, and it takes you 1 hour to get there, what is your average velocity? If we label West Point as our origin or start point, and we begin our trip at *time* = 0, our average velocity is (see page 85).

$$v_{average} = \frac{\text{change in displacement}}{\text{change in time}} = \frac{\Delta d}{\Delta t} = \frac{50\,miles - 0\,miles}{1\,hour - 0\,hour} = 50\,mph$$

(A note about the notation here: "Δ" is the Greek capital letter "delta" which means "change in". So, the symbol Δx means "change in the quantity x"). Using the above equation, we can calculate our average velocity over any distance and time interval. But what if we what to know our velocity at a single instant in time or, in other words, our instantaneous velocity? How would we go about finding a reasonable answer? One way would be to find the average velocities over increasingly smaller time intervals, so our change in time, or Δt, would be getting smaller and smaller and would approach zero. Mathematically, we would write this as $\Delta t \to 0$, and define the

instantaneous velocity as the limiting value (you should be familiar with the notion of a limit from MA103) of these average velocities over shorter and shorter time intervals.

Velocity is certainly not the only rate of change, so in general we can say that in order to find the instantaneous rate of change for any function, we find the limiting value of the average change in the function $y = f(x)$ as the change in x gets smaller and smaller, or $\Delta x \to 0$. Written mathematically, this would look like the following: $\text{instantaneous change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

To cement the importance of this relationship between average and instantaneous change, and to motivate our study of the derivative, we can now formulate the most important idea in Block I: **Instantaneous rate of change is the limiting value of the average rate of change, and instantaneous rate of change is precisely what a derivative is.** The last statement is crucial to your understanding of differential calculus and to your success in this course.

Note: A superficial understanding of the derivative will severely limit your ability to solve complex problems in a single variable setting, and using the derivatives of multivariate functions to model and solve problems will simply be beyond your reach.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 2 – LIMIT OF A FUNCTION (REVIEW)

OBJECTIVES:

1. Understand what it means for a function to have a limit.
2. Approximate the limit of a function using graphic and numeric means.
3. Approximate the left and right-hand limits of a function.
4. Understand what it means for a function to have an infinite limit.
5. Be able to determine when a function decreases/increases without bound.

READ:

1. Stewart: Section 2.2, pages 88-96.
2. Student Notes.

THINK ABOUT:

1. What does it mean for an approximation to be "good enough"?
2. When approximating a quantity, how do we determine if our approximation is sufficient?
3. How are infinite limits and vertical asymptotes related?

DO: Section 2.2/ 3, 4, 7, 8, 13, 19, 25

STUDENT NOTES:

Sequences, like limits, are also a topic that you should be familiar with from MA103. Your study of recursion equations should have led you to the conclusion that sequences and recursion equations are one and the same, and that exploring the long term behavior of a recursion equation is analogous to finding the limiting value of a sequence (which may or may not exist). This lesson will serve to both reinforce and extend your knowledge of limits in order that you may better understand the notion of instantaneous change, which as we know from lesson one, is the limiting value of the average rate of change of the function as the change in our dependent variable becomes smaller and smaller, and is of fundamental importance in understanding the derivative.

Definition 1 on page 88, is foundational to understanding limits. The notation that is used here is:

$$\lim_{x \to a} f(x) = L$$

The book explains how to read this expression, but the meaning of it can be difficult to grasp. We start with an x coordinate of a. The definition says that as we travel along a function and get closer and closer to a on the x-axis (from both the right and the left sides) then we are approaching a y value. From this definition, that y value is some number L. Look at Figure 2 (a, b, and c) on page 89 for a graphical representation of what this means.

Focus your attention on examples 1 & 2. These show what a limit looks like both graphically and numerically (in a table of values).

In Definition 1, we looked at the limit as x approached a from both sides. We can also consider just one side of the limit, such as what is written in Definition 2 on page 93:

$$\lim_{x \to a^-} f(x) = L$$

This is the limit as *x* approaches *a* from values less than *a* (i.e. as *x* approaches *a* from the left). The opposite would be the limit as *x* approaches *a* from values greater than *a* (i.e. as *x* approaches *a* from the right). This would be written as:

$$\lim_{x \to a^+} f(x) = L$$

Finally, Definition 3 on page 93 shows what is necessary for there to be an overall limit at some point *a*. What is required is that there exist a left-hand limit and a right-hand limit and these two limits must be equal. We write this as:

$$\lim_{x \to a} f(x) = L \quad \text{IF AND ONLY IF,} \quad \lim_{x \to a^-} f(x) = L \quad \text{AND} \quad \lim_{x \to a^+} f(x) = L\,.$$

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 3 – CONTINUITY (REVIEW)

OBJECTIVES:

1. Understand the mathematical and geometric definitions of continuity.
2. Determine if a function is continuous, both at a point and on an interval.
3. Understand the three different types of discontinuities at a point: removable, jump, and infinite.
4. Understand the Intermediate Value Theorem and its applications.

READ:

1. Stewart: Section 2.5, pages 119-127.
2. Student Notes.

THINK ABOUT:

1. Why is it important for a function to be continuous?
2. What is the difference between a function being continuous at a point and a function being continuous on an interval?
3. What is an example of a discontinuous quantity in your life?
4. What is useful about the Intermediate Value Theorem? How can we use it?

DO: Section 2.5/ 3, 4, 7, 8, 16, 37, 47

STUDENT NOTES:

Definition 1 on page 119 gives the mathematical requirements for a function to be continuous at a point (or as the book says at some number *a*). We read the definition of $\lim_{x \to a} f(x) = f(a)$ as a set of three conditions that must be met in order for a function to be continuous at a point.

1. First, $f(a)$ must be defined (i.e. there is no *hole* in the function at this point).
2. Next, in order for the definition there must be a limit at this point (i.e. from *both sides* the function must approach the same limit as ***x*** approaches ***a***).
3. Finally, the *answers* to 1 & 2 must equal. Therefore, the limit that the function approaches and the function value at the point must be the same.

If any of these three conditions are not met then we say that the function is ***discontinuous at a.***

Focus your attention on understanding Example 2 and ensure that you not only understand continuity, but are also able to determine when an equation is continuous and when it is not.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 4 – DERIVATIVES AND RATES OF CHANGE (REVIEW)

OBJECTIVES:

1. Find the slope of a line tangent to a curve at a point using the limiting value of the slopes of the secant lines.
2. Understand the relationship between average rate of change and instantaneous rate of change graphically, numerically, and algebraically.
3. Know and understand the definition of the derivative at a number.

READ:

1. Stewart: Section 2.7, pages 143-150.
2. Student Notes.

THINK ABOUT: Given the graph of a function, how would you approximate the instantaneous rate of change of the function at a point?

DO: Section 2.7/ 3, 13, 16, 22, 41

STUDENT NOTES:

The main definition for you to learn is Definition 1 on page 144 (Equation 2 and Definition 4 are important, but realize that it these are just a different way of writing the same basic definition). Definition 1 gives what is needed for the equation of a tangent line. To understand this definition, we need to remember that a tangent line can be approximated by a secant line (remember a ***secant line*** is a line crossing the function in two points). What we do with these two points (here labeled ***x*** and ***a***) is we bring them so close together that they are *practically on top of each other*. Therefore, we say that ***x*** is approaching ***a*** and we write it as:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

On page 145, another expression is given for the slope of a tangent line. Instead of using the notation of ***x*** and ***a*** we will instead use the variable ***h***. We will let ***h*** be the distance between ***x*** and ***a*** which we can write in the form of an equation as: $h = x - a$. We can see that as $x \to a$, then h will get closer and closer to zero. Some simple algebra allows us to rearrange this equation as $x = h + a$. Using these new equations, and Definition 1, we can re-write the equation for the slope of the tangent line (which is the definition of the derivative at a point) as:

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{a + h - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

Starting on page 146, the book uses this definition to define the derivative more generally as the instantaneous rate of change of any function *f* at the point *a*. Examples 3, 6, and 7 use the derivative to describe velocity (Example 3), the rate of change of production costs (Example 6), and the rate at which the national debt is changing (Example 7).

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 5 – THE DERIVATIVE AS A FUNCTION

OBJECTIVES:

1. Know and understand the definition of the derivative and apply the definition to find derivatives of basic polynomial functions.
2. Understand when a function fails to be differentiable.
3. Understand the graphical, algebraic and physical interpretations of the derivative.

READ:

1. Stewart: Section 2.8, pages 154-161.
2. Student Notes.

THINK ABOUT: Why do we have different notations for representing the derivative?

DO: Section 2.8/ 17, 20, 21, 35, 38

STUDENT NOTES:

In Section 2.8, we begin to look at the derivative as something that is not just derived from the function f but that is itself a function. As a function, we can look at the derivative at different points, and we can discern things about the original function f by looking at its change (f'). In Figure 2 on page 155 we see the graph of the function on top and its derivative below it.

Starting on page 157, skim the remaining material in Section 2.8. Focus your attention on the following: (a) Other notation for the derivatives, (b) when a function is and is not differentiable, and (c) how a function can fail to be differentiable. Skim through the examples on pages 157-161.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 6 – DERIVATIVES OF POWER, EXPONENTIAL, AND LOG FUNCTIONS

OBJECTIVES:

1. Apply the Constant Multiple, Sum and Difference Rules to calculate derivatives of algebraic functions.
2. Calculate the derivative of a power function using the Power Rule.
3. Calculate the derivative of the natural exponential function.
4. Calculate the derivative of the natural log function.

READ:

1. Stewart: Section 3.1, pages 173-180.
2. Student Notes.

THINK ABOUT: What is unique about the natural exponential function?

DO: 1. Section 3.1/ 3, 5, 7, 9, 11, 13, 15, 17, 19, 22, 32, 45

2. Find the derivatives of the following functions:

$y(x) = 3e^x$

$f(t) = \ln(t)$

STUDENT NOTES:

Chapter 3 is on the mechanics of taking the derivative. Up to this point, we have spent a lot of time trying to understand what a derivative is and what its interpretation is. Now we begin to actually take derivatives.

In addition to the power and exponential rule in Section 3.1, you must know the derivative of the natural logarithm function. The following is the derivative for this function.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

You should realize that the best way to learn the rules of differentiation is through lots of practice. Each section in this chapter has numerous "DO" problems; this is intentionally designed to *encourage* you to work through as many types of problems as possible to get comfortable with actually using the many differentiation laws.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 7 – PRODUCT AND QUOTIENT RULES

OBJECTIVES:

1. Calculate derivatives using the Product Rule.
2. Calculate derivatives using the Quotient Rule.

READ:

1. Stewart: Section 3.2, pages 183-187.
2. Student Notes.

THINK ABOUT: Why **doesn't** $\frac{d}{dx} f(x)g(x)=f'(x)g'(x)$?

DO:

1. Section 3.2/ 1, 4, 5, 7, 26, 27, 33, 34
2. Section 3.6/ 10, 16

STUDENT NOTES:

A more detailed example of both the product and quotient rule are given below in order to help you better understand how to do these types of problems.

Example 1: $\begin{array}{l} Given: \ h(x) = x^2(x^3+2x) \\ Find: \ h'(x) \end{array}$

First, we need to identify that this is a product rule problem (i.e. two functions multiplied together). Therefore, it may be helpful to write down that the two functions are:

$$f(x) = x^2$$
$$g(x) = (x^3+2x)$$

The product rule on page 184 says the following:

$$h'(x) = \frac{d}{dx}[f(x)g(x)] = f(x)\square g'(x) + f'(x)\square g(x).$$

Once again, it may be helpful to take the derivatives of the two functions separately and then substitute them back into the product rule formula.

$$f(x) = x^2 \qquad f'(x) = 2x$$
$$g(x) = (x^3+2x) \qquad g'(x) = 3x^2+2$$

$$h'(x) = \frac{d}{dx}[f(x)g(x)] = f(x)\square g'(x) + f'(x)\square g(x)$$

$$h'(x) = (x^2)(3x^2+2) + (2x)(x^3+2x)$$

Example 2: $Given: \; h(x) = \dfrac{x^2}{(x^3 + 2x)}$

$Find: \; h'(x)$

First, we need to identify that this is a quotient rule problem (i.e. one function divided by another.) Again, the two functions are:

$$f(x) = x^2$$
$$g(x) = (x^3 + 2x)$$

The quotient rule on page 186 says the following:

$$h'(x) = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}.$$

Using the derivatives of the two functions, we will once again substitute them back into the quotient rule formula.

$$f(x) = x^2 \qquad f'(x) = 2x$$
$$g(x) = (x^3 + 2x) \qquad g'(x) = 3x^2 + 2$$

$$h'(x) = \frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}$$
$$= \frac{(x^3 + 2x)(2x) - (x^2)(3x^2 + 2)}{((x^3 + 2x))^2}$$

<u>Links to Other Pages</u>: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 8 – DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

OBJECTIVES:

1. Be able to calculate the derivatives of the six trigonometric functions.

READ:

1. Stewart: Section 3.3, pages 189-195.
2. Student Notes.

DO: Section 3.3/ 1, 3, 6, 7, 10, 13, 14, 23, 31

STUDENT NOTES:

One note (which is often missed), is right above the red box of *Derivatives of Trigonometric Functions* (page 193), is ***"Remember that they are valid only when x is measured in radians."*** So you need to make sure that your calculator is in radians mode (*Mathematica* defaults to radians).

The two primary derivatives that you must learn (memorize) are:

$$\frac{d}{dx}(\sin x) = \cos x \text{ and}$$
$$\frac{d}{dx}(\cos x) = -\sin x$$

The remaining derivatives on page 193 can all be derived by an application of the product or quotient rule of these two derivatives. The other option is to simply memorize the six different trigonometric derivatives. Below is an example of how to find the derivative of the trigonometric function without a lot of memorization.

Example: $h(x) = \tan(x)$, find $h'(x)$. First we need to remember that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, therefore our function $h(x) = \dfrac{\sin(x)}{\cos(x)}$. This is a quotient of two functions, so we need to apply the quotient rule. **Note:** $\sin(x) = \sin x \neq \sin \times x$**.**

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$
$$\frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right] = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

Remembering once again our basic geometry:

$$\frac{1}{\cos^2(x)} = \left(\frac{1}{\cos(x)}\right)^2 = (\sec(x))^2 = \sec^2(x).$$

We can verify this by looking at the chart on page 193 which gives: $\frac{d}{dx}(\tan x) = \sec^2 x$.

<u>Links to Other Pages</u>: Block I: Calendar and Overview Block I: Endstate Problems

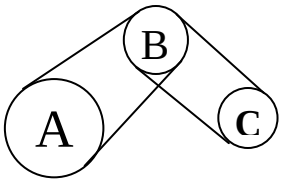
LESSON 9 – THE CHAIN RULE

OBJECTIVES:

1. Calculate derivatives of composite functions using the chain rule.

READ: Stewart: Section 3.4, pages 197-202 (Stop before *How to Prove the Chain Rule*, pg 202).

THINK ABOUT: Can you interpret the chain rule in terms of the pulley system below? If we rotate (change) A slightly, how does C rotate (change), or in other words, what is the rate of change of C with respect to A?



DO:

1. Section 3.4/ 1, 5, 7, 9, 11, 13, 25, 53
2. Section 3.6/ 3, 4, 9, 23

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 10 – Course Wide Derivative WRIT

NOTE: Attendance at WRIT is mandatory. Cadets will not make any appointments, except in emergencies, that would preclude attendance. **CADETS ARE REQUIRED TO NOTIFY THEIR INSTRUCTOR IN ADVANCE OF ANY WRIT ABSENCE**.

OBJECTIVE:
The Course Wide Derivative WRIT will assess your knowledge specifically of lessons 5-9 and will be given during class. You are required to achieve an 80% or higher on the WRIT this or you will retake the test until you earn a score of 80%. Furthermore, the grade that you earn on this WRIT will count toward your final course grade.

REVIEW: Previous lessons as needed.

HOMEWORK: The CW Derivative Homework is due at the beginning of class.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 11 – MAXIMUM/MINIMUM VALUES OF FUNCTIONS

OBJECTIVES:

1. Know the definitions of local (or relative) and absolute (or global) extreme values of a single variable function.
2. Understand the definition of a critical number and be able to find the critical numbers of a single variable function.
3. Understand Fermat's Theorem and be able to say why its converse might be false.
4. Understand the Extreme Value Theorem and its applications.
5. Be able to apply the Closed Interval Method for a continuous function on a closed interval in order to find the absolute maximum and minimum:
   a. Determine the function value at the critical numbers.
   b. Determine functional values at endpoints of the interval.
   c. Classify the largest values from *a* and *b* above as the absolute maximum, and the smallest from *a* and *b* as the absolute minimum.

READ:

1. Stewart: Section 4.1, pages 271-276.
2. Student Notes.

THINK ABOUT: In Figure 6 on page 272, why isn't the point (1,3) the absolute maximum? Why does the text say there isn't one at all?

DO: Section 4.1/ 3, 6, 31, 43, 51, 60

Following Problem:

$$f(t) = \cos(10t)$$

find $\dfrac{df}{dt}$

STUDENT NOTES:

Many of the definitions found in this section are the key to understanding both max/min problems, as well as optimization problems which we will get to in a few lessons. Definition 1 on page 271 states that a function has an absolute maximum at a point *c* if the function value (written as $f(c)$) is larger than anywhere else. Graphically, this would be the highest point on the graph. An absolute minimum is simply the point with the lowest function value (or the low point on the graph). Definition 2 on page 271 applies the same idea only it's called a local max or local min. The only difference is that a local max/min value is found when looking at a smaller section of the function (i.e. values that are *local* to the point we are considering). We can have numerous local max and min values (as an example a sine or cosine graph has multiple local max and min points – Example 1) but the Extreme Value Theorem on page 272 implies that we can only have one absolute max and one absolute min value for any function, when we consider a closed interval (i.e. a section of a function with endpoints).

For both the max/min problems as well as optimization, it is extremely important that you understand how to find the critical numbers of a function. Definition 6 on page 274 defines critical numbers and Example 7 shows how to find them. Finally, the Closed Interval Method on

page 275 is a step by step process for finding the absolute max and min values of a function on a closed interval. Be sure to understand that if we have an absolute maximum at some number *c*, we are referring to the *x*-value, while the absolute maximum function value is $f(c)$, or the *y* value. The critical point would therefore be $(c, f(c))$.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 12 – DERIVATIVES AND THE SHAPES OF CURVES

OBJECTIVES:

1. Be able to classify critical points as local maxima or minima using:
   a. The First Derivative Test (Examining increasing/decreasing nature of function).
   b. The Second Derivative Test (Examining function concavity).
   c. A plot of the function.
2. Given the graph of a function, sketch its first and second derivative functions.
3. Given the graph of the derivative of a function, sketch a possible original function.

READ: Stewart: Section 4.3, pages 287-294.

THINK ABOUT:

1. What does a function's derivative tell us about the function?
2. What does a function's second derivative tell us about the function?

DO: Section 4.3 / 2, 8, 19, 22, 32
Section 4.1 / 69
Following Problem:

$$v(t) = \frac{3t^2}{(t^3+1)}$$

find $v'(t)$

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 13 – SOLVING OPTIMIZATION PROBLEMS I

OBJECTIVES:

1. Model and solve optimization problems.
   a. Given a problem, be able to determine a single or multivariable objective function.
   b. Given limitations in a problem, write a constraint equation.
   c. Transform a multivariable objective function into a single variable objective function when only one constraint exists.
2. Reinforce understanding of the problem solving technique and apply it to optimization problems such as (but not limited to):
   a. Minimize materials given an area or volume constraint.
   b. Maximize area or volume given a material constraint.
   c. Minimize cost.
   d. Minimize distance or cost given physical layout constraints.
   e. Other various applications as introduced by your instructor.

READ:

1. Stewart: Section 4.7, pages 322-327.
1. Student Notes.

THINK ABOUT:

1. Included in the Student Course Guide is a seven step process. How do these seven steps relate to the Polya Problem Solving Process? (Note: Stewart references Polya both explicitly and implicitly as described on page *xiv* in the preface of your text.)
2. At which step would it be best to check your work using *Mathematica*? Specifically, what would you graph with it?
3. In Example 2 on page 323, why do we need to check that the value obtained is an absolute minimum?
4. In all the examples, how is the maximum or minimum value verified (there are numerous answers to this question)?

DO: Section 4.7/ 3, 6, 13, 19

Following Problem:

$$h(x) = y^3 + \pi^2 + x^2$$

find $\dfrac{dh}{dx}$

STUDENT NOTES:

Focus your attention on page 322 of the text. Stewart develops a process for solving optimization problems which fits in nicely with Polya's Process. Recall that Polya listed four steps in his process: understand the problem, develop a plan, carry out the plan, and look back. Though these steps are good, sometimes students become confused at what is being asked.

Additionally, the text never closes the loop and talks about answering the question. Below is a slightly modified way to look at the process of solving optimization problems.

Step	Title	Description	Polya	USMA
1	Read & Understand the Problem	This is the most important step of the process. Look for key words to signify a min or max. What information can you	Understand the Problem	Transform
2	Draw a Picture/Diagram	If you can draw a picture to help, then do it. This will help you understand the problem		
3	Define variables	These are items that you need to determine in the problem and can be often just labeled on the picture. Include units if provided in problem		
4	Define Objective Function and Constraint Equations (if any)	Define Objective Function (what we min/max) in terms of the variables from step 3. If there are more than one variable, then we have to reduce it using a constraint equation. This is something that is equal to something else in the problem. If an equal sign is not provided look for key words.	Develop A Plan	
5	Simplify Objective Function (May not be required)	If you have more than one variable in your objective function, then you must use constraint equations to reduce the number to one.		
6	Solve the problem	This step has all the "calculus" involved. These problems require you do one of two things: if it is a boundary value problem then you will used the Closed Interval Method and if not, then you will solve for a local extrema and verify using some appropriate extrema test.	Carry out the Plan	Solve
7	Answer the Question	The step is not included back in the Stewart text on pg 322, but it should be obvious. Use the information from Step 6 to answer the question. Include units when appropriate. Does your answer make sense in the context of the problem?	Look Back	Interpret

Following the process above will help you organize your thoughts better and lead you on the path to success.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 14 – SOLVING OPTIMIZATION PROBLEMS II

OBJECTIVES:

1. Model and solve optimization problems.

READ: Stewart: Section 4.7, pages 322-327.

DO: Section 4.7/ 12, 24, 31, 33
Following Problem:

Optimization Problem Requiring the Closed Interval Method

A manufacturer can produce at most 120 units of a certain product each year. The demand equation for the product is given by $p = x^2 - 100x + 3200$, where x is the number of units that can be sold each year at a price of p dollars. The manufacturer's average cost function is

$$\overline{C}(x) = \frac{2}{3}x^2 - 40x + \frac{10{,}000}{x},$$

where $\overline{C}$ is the average cost per unit (in dollars) required to produce x units in a given year. How many units should be produced in order to maximize profit?[3]

Following Problem:

$g(x) = (x^3)(10 - 4x^2 + 3x)$

find $g'(x)$

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

[3] Problem developed by Dr Edward W. Swim, Assistant Professor Department of Mathematical Sciences, United States Military Academy, June 11, 2008.

LESSON 15 – SOLVING OPTIMIZATION PROBLEMS III

OBJECTIVES:

1. Model and solve optimization problems.

READ: Stewart: Section 4.7, pages 322-327.

DO: Section 4.7/ 27, 30, 33, 36, 38
Following Problem:

Optimization Problem Requiring the First Derivative Test

A quality loss function of the form $L(t,x)=\frac{1}{4}(x^4+t^4)-xt(x^2+t^2)+\frac{3}{2}x^2t^2$ is a financial measure of user dissatisfaction with a product's performance as it deviates from a target value. For example, if a manufacturer of automobile pistons advertises a new model will be *t* cm in diameter, then the quality loss function measures how consumer dissatisfaction with the new model will change if the actual diameter is *x* cm. Find the actual diameter necessary to minimize customer dissatisfaction if the manufacturer advertises that the piston has a diameter of 5 cm.[4]

Following Problem:

$f(x)=e^{2x+3}$

find $f'(x)$

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

[4] Problem developed by Dr Edward W. Swim, Assistant Professor Department of Mathematical Sciences, United States Military Academy, June 11, 2008.

LESSON 16 – PROBLEM SOLVING LAB 1

OBJECTIVE:

Solve problems by applying concepts relevant to the current material. Your instructor will provide you guidance for this lesson.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 17 – IMPLICIT DIFFERENTIATION

OBJECTIVES:

1. Understand what is meant by implicitly and explicitly defined functions.
2. Use the method of implicit differentiation to find the derivative of implicitly defined functions.

READ:

1. Stewart: Section 3.5, pages 207-211 (Stop at *Derivatives of Inverse Trig Functions*).
2. Student Notes.

THINK ABOUT: Why is it sometimes necessary to use implicit differentiation?

DO: Section 3.5/ 1, 5, 9, 10, 25, 38
Following Problem:

$f(x) = (x-1)e^x$

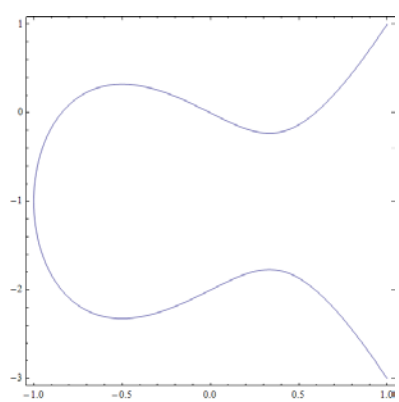
find $f''(x)$

STUDENT NOTES:

It is often useful to graph a function before we begin to "do" anything with it. We should all be able to define and graph an explicitly defined function in *Mathematica*, but how do we go about plotting a function that is defined implicitly? No need to worry, *Mathematica* makes this quite easy for us. We simply use the command, "ContourPlot", define our function, and then set the bounds for which we want it graphed.

For instance, if we want to plot the implicit function $y^2 + 2y = 4x^3 + x^2 - 2x$, we would execute the following command in *Mathematica* exactly as it is written here (note: "control 6" is the shortcut key in *Mathematica* to raise a value to a power):

```
ContourPlot[y^2 + 2 y == 4 x^3 + x^2 - 2 x, {x, -1, 1}, {y, -3, 1}]
```



Sometimes functions such as $y^2 + 2y = 4x^3 + x^2 - 2x$ are not as easy to solve for y explicitly as a function of x by hand as other functions. Fortunately, we don't always need to

solve equations like these for *y* in terms of *x* in order to find $\frac{dy}{dx}$. To implicitly differentiate a function, remember to first differentiate both sides of the equation with respect to one variable. In this example, we differentiate with respect to *x*. Then solve the resulting derivative for $\frac{dy}{dx}$ be rearrange the terms as shown.

$$y^2 + 2y = 4x^3 + x^2 - 2x$$

remember that the variable y is in terms of x

$$(y(x))^2 + 2(y(x)) = 4x^3 + x^2 - 2x$$

using the chain rule

$$2y\frac{dy}{dx} + 2\frac{dy}{dx} = 3(4x^2) + 2x - 2$$

now isolate $\frac{dy}{dx}$

$$\frac{dy}{dx}(2y+2) = 12x^2 + 2x - 2$$

$$\frac{dy}{dx} = \frac{12x^2 + 2x - 2}{2y+2}$$

$$\frac{dy}{dx} = \frac{12x^2 + 2x - 2}{2(y+1)}$$

$$\frac{dy}{dx} = \frac{6x^2 + 1x - 1}{(y+1)}$$

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 18 – PROBLEM SOLVING WITH RELATED RATES I

OBJECTIVES:

1. Model and solve problems using relationships between rates of change. These are normally referred to as related rates problems. Some of the common types of problems are:
    a. Changing surface area problems
    b. Changing volume problems
    a. Pythagorean Theorem problems
    b. Similar triangles problems
    c. Changing angle problems
2. Model and solve complicated scenarios using relationships between rates of change More challenging problems include scenarios that are not modeled by geometric shapes. Examples of these may include, but are not limited to:
    a. Changing supply and/or changing demand problems
    b. Changing production problems

READ:

1. Stewart: Section 3.9, pages 241-245.
2. Student Notes.

THINK ABOUT:

1. How can dimensional analysis assist in setting up a problem that relates rates of change?
2. How will your knowledge of trigonometry help with setup up a problem that relates rates of change?
3. How can a good diagram or picture assist in setting up a problem that relates rates of change?
4. What are two examples in your set of experiences where things change with time and affect other things?

DO: Section 3.9/ 2, 5, 7, 11, 12

Following Problem:

$$f(t) = \sin(25t)$$

find $\dfrac{df}{dt}$

STUDENT NOTES:

Focus your attention on page 243 of the text. Here Stewart develops a process for solving related rates problems, similar to what he did in the section on optimization. Likewise, on the following page is a seven step table similar to the one introduce for optimization. You can notice many similarities in this problem solving approach.

Step	Title	Description	Polya	USMA
1	Read & Understand the Problem	This is the most important step of the process. Look for key words to signify a related rates problem.	Understand the Problem	Transform
2	Draw a Picture/Diagram	If you can draw a picture to help, then do it. This will help you understand the problem.		
3	Define variables	These are items that you need to determine in the problem and can be often just labeled on the picture. Include units if provided in problem. Time will always be one of the variables.		
4	Determine Quantities that are Changing	These are the variables that you know are changing with respect to time and the variable or quantities that you are trying to determine that are changing with respect to time.		
5	Define your "Relationship" Equation	Based on your information from Step 4, you need to determine an equation that relates the changing rates (both known and unknown) together in order to be able to solve the problem.	Develop A Plan	
6	Take the Derivative with respect to Time	This step has all the "calculus" involved. This is where you must take the derivative with respect to time by using the chain rule. *Head the warning on Page 243 in the Stewart textbook about substituting values into the relationship equation too early. If the quantity changes, then substitute the value during Step 7.*	Carry out the Plan	Solve
7	Answer the Question	This is the final step. Here you substitute all your known quantities into the result from Step 6 and solve for any unknowns. Remember, the result of this process might not answer the question, so ensure that you answer the question. Does your answer make sense? What are the units of the rate that you found? Does a quick dimensional analysis check out with the units of your answer?	Look Back	Interpret

Again pay close attention to the warning at the bottom of page 243. It describes a very common error of substituting the given numerical information too early. Why do you think this is a problem? Well, if we substitute in a number or constant for a quantity that actually varies with respect to time and then take the derivative, we get zero since the derivative of any constant, as we all know, is zero. Only make the proper substitution after you have taken the derivative of the Relationship Equation developed in Step 5.

NOTE: Stewart, as many other introductory calculus books, incorporates geometry into the set up of most of his related rates problems. This is done because problem solving with related rates is more difficult for students to grasp then other calculus topics up to this point. As such, Stewart uses students' knowledge of geometric formulas, such as the Pythagorean theorem, or the equation of a circle, as a launching platform to help facilitate the introduction of related rates.

However, you as the student should also be comfortable with applying implicit differentiation to solve related rates type problems to any function, such as the one in the following example:

EXAMPLE: An employment research company estimates that the value of a recent MBA graduate to an accounting company is $V = 3e^2 + 5g^3$ where V is the value of the graduate, e is the number of years of prior business experience, and g is the graduate school grade-point average (GPA). A company that currently employs graduates with a 3.0 GPA wishes to maintain a constant employee value of $V = 200$ but finds that the GPA of is new employees is dropping at a rate of 0.2 per year. How fast must the experience of its new employees be growing in order to compensate for the decline in GPA[5]?

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems.

[5] Stefan Waner and Steven R. Costenoble, *Applied Calculus 3rd edition*. Thomson Brooks/Cole 2004. pp 335.

LESSON 19 – PROBLEM SOLVING WITH RELATED RATES II

OBJECTIVES:

Continue to model and solve related rates problems.

READ: Stewart: Section 3.9, pages 241-245.

DO: Section 3.9/ 15, 18, 23, 27, 31, 33
Following Problem:

$$v(t) = \frac{3t^3}{(t^4+1)}$$

find $v'(t)$

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 20 – MATHEMATICAL DIMENSIONS

OBJECTIVES:

1. Understand the 3-D coordinate system and identify the coordinate planes within the system.
2. Determine the distance between points in 3 dimensional space. (This lesson objective will not be covered in class. Students are encouraged to learn this lesson objective on their own as they will see applications requiring knowledge of this objective in future lessons. Students will also be assessed on their understanding of this objective in future classroom evaluations.)
3. Understand how to project a point onto a plane.

READ: Stewart: Section 12.1, pages 765-768.

THINK ABOUT: See if you can derive the distance formula on page 767 (look at Figure 9).

DO: Section 12.1/ 3, 4, 6, 8, 10
Following Problem:

$$v(t) = \frac{3t^2}{(t^3+1)}$$

find $v'(t)$

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 21 – PROBLEM SOLVING LAB 2

OBJECTIVE:

Solve problems by applying concepts relevant to the current material. Your instructor will provide you guidance for this lesson.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 22 – PROJECT RESEARCH DAY

OBJECTIVE:

Solve problems by applying concepts relevant to the current material. Your instructor will provide you guidance for this lesson.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 23 – WRITTEN PARTIAL REVIEW I

NOTE: Attendance at WPRs is mandatory. Cadets will not make any appointments, except in emergencies, that would preclude attendance. **CADETS ARE REQUIRED TO NOTIFY THEIR INSTRUCTOR IN ADVANCE OF ANY WPR ABSENCE**.

WPR IS DURING DEAN'S HOUR

OBJECTIVE:

WPR I will assess your knowledge of Lessons 1-21 and will be given during Dean's hour on 12 February 2009. This WPR will be given in two parts: a non-technology part and a technology part. Initially, you will be given the entire exam but will only be authorized your issued calculator. Upon you completion of the non-technology portion of the exam, you will turn it in. Once the non-technology portion is submitted, you will be authorized the use of your laptop computer (wireless disconnected) and a sheet of hand written notes to complete the technology portion. Additionally instructions will provided by your instructor.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems

LESSON 24 – WRITTEN PARTIAL REVIEW I AAR

OBJECTIVE:

This lesson is set aside for conducting an extensive After Action Review of your performance on WPR I. After this lesson you should go back and resolve all problems that you had on the test. This will help you start studying for the Term End Exam.

Links to Other Pages: Block I: Calendar and Overview Block I: Endstate Problems