MA205 - Integral Calculus
Lesson 47: Exponential Growth and Decay

Problem Solving Problems

1. A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After 3 hours there are 8000 bacteria.

   (a) Model the growth of the bacteria with a differential equation.

   (b) Find the number of bacteria after 4 hours.

   (c) Find the rate of growth after 4 hours.

   (d) When will the population reach 30,000?

2. The table gives the population of the United States, in millions, for the years 1900-2000.

Year	Population	Year	Population
1900	76	1960	179
1910	92	1970	203
1920	106	1980	227
1930	123	1990	250
1940	131	2000	275
1950	150		

(a) Use the exponential model and the census figures for 1900 and 1910 to predict the population in 2000. Compare with the actual figures as recorded and try to explain the discrepancy.

(b) Use the exponential model and the census figures for 1980 and 1990 to predict the population in 2000. Compare with the actual population. Then use this model to predict the population in the years 2010 and 2020.

(c) Graph both of the exponential functions in parts (a) and (b) together with a plot of the actual population. Are these models reasonable ones? Why or why not?

3. Experiments show that if the chemical reaction

$$N_2O_5 \longrightarrow 2NO_2 + \frac{1}{2}O_2$$

takes place at 45°C, the rate of reaction of dinitrogen pent-oxide is proportional to its concentration as follows:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5]$$

Using Example 4 in Section 3.3 as a guide

(a) Find an expression for the concentration $[N_2O_5]$ after t seconds if the initial concentration is C.

(b) How long will the reaction take to reduce the concentration of N_2O_5 to 90% of its original value?

4. The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.

(a) Find the mass that remains after t years.

(b) How much of the sample remains after 100 years?

(c) After how long will only 1-mg remain?

5. Scientist can determine the age of ancient objects by a method called *raidio-carbon dating.* the bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope of carbon, ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore, the level of radioactivity must also decay exponentially. A parchment fragment was discovered that had about 74% as much ^{14}C radioactivity as does plant material on Earth today. Estimate the age of the parchment.

6. A roast turkey is taken from an oven when its temperature has reached 185°F and is placed on a table in a room where the temperature is 75°F.

   (a) If the temperature of the turkey is 150°F after half an hour, what is the temperature after 45 min?

   (b) When will the turkey cool to 100°F?

7. Consider a population $P = P(t)$ with constant relative birth and death rates α and β, respectively, and a constant emigration rate m, where α, β, and m are positive constants. Assume that $\alpha > \beta$. Then the rate of change of the population at time t is modeled by the differential equation

$$\frac{dP}{dt} = kP - m \qquad k = \alpha - \beta$$

(a) Find the solution of this equation that satisfies the initial condition $P(0) = P_0$.

(b) What condition on m will lead to an exponential expansion of the population?

(c) What condition on m will lead to a constant population? A population decrease?

(d) In 1847, the population of Ireland was about 8 million and the difference between the relative birth and death rates was 1.6% of the population. Because of the potato famine in the 1840s and 1850s, about 210,000 inhabitants per year emigrated from Ireland. Was the population expanding or declining at that time? Justify your conclusion.

8. If \$500 is borrowed at 14% interest

   (a) Find the amounts due at the end of 2 years if the interest is compounded (i) annually, (ii) quarterly, (iii) monthly, (iv) daily, (v) hourly, and (vi) continuously.

   (b) Suppose \$500 is borrowed and the interest is compounded continuously. If $A(t)$ is the amount due after t years, where $0 \le t \le 2$, graph $A(t)$ for each of the interest rates 14%, 10%, and 6% on a common screen.