Lesson 43 - Applications - The Logistics Equation

Objectives

- Model and solve situations involving population growth using the logistic equation.
- Describe long term behavior.
- Identify equilibrium solutions.

READ

- Stewart, Chapter 9.4, pages 592-597.

MATHEMATICA COMMANDS AND TASKS YOU NEED TO KNOW

No new commands.

Problem Solving Problems

1. Suppose a population develops according to the logistic equation

$$\frac{dP}{dt} = 0.05P - 0.0005P^2$$

where t is measured in weeks.

(a) What is the carrying capacity? What is the value of k?

Carrying capacity $\Rightarrow 100$

$k = 0.05$

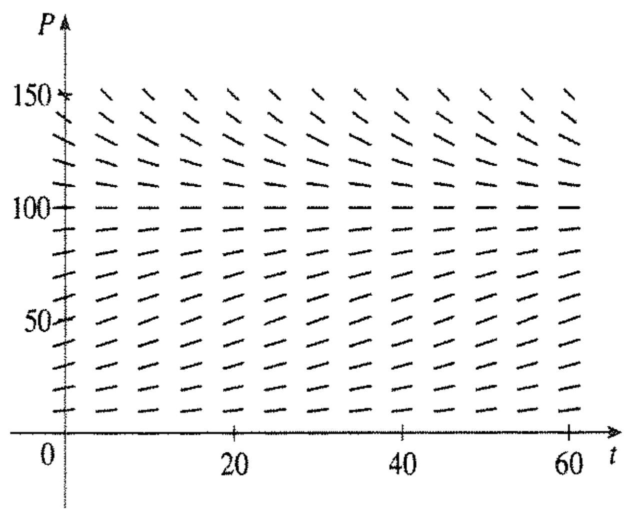
(b) A direction field for this equation is shown below. Where is the rate of change close to 0? Where is the rate of change the largest? Which solutions are increasing? Which solutions are decreasing?

$\frac{dP}{dt} \approx 0 \Rightarrow P = 0$, $P = 100$

$\frac{dP}{dt}$ largest $\Rightarrow P = 50$

$\frac{dP}{dt} > 0 \Rightarrow 0 < P < 100$

$\frac{dP}{dt} < 0 \Rightarrow P > 100$



(c) Use the direction field to sketch solutions for initial populations of 20, 40, 60, 80, 120, and 140. What do these solutions have in common? How do they differ? Which solutions have inflection points? At what population levels do they occur?

Converge to $P = 100$

$P = 20$ & 40 have inflection points ($P = 50$)

(d) What are the equilibrium solutions? How are other solutions related to these solutions?

$P = 0$, $P = 100$

2. Suppose that a population grows according to a logistic model with carrying capacity 6000 and $k = 0.0015$ per year.

   (a) Write the logistic differential equation for these data?

   $$\frac{dP}{dt} = 0.0015P\left(1 - \frac{P}{6000}\right)$$

   (b) Draw a direction field (either by hand or with Mathematica). What does it tell you about the solution curves?

   (c) Use the direction field to sketch the solution curves for initial populations of 1000, 2000, 4000, and 8000. What can you say about the concavity of these curves? What is the significance of the inflection points?

   $\left.\begin{array}{l} P=1000 \\ P=2000 \end{array}\right\}$ Concave up until P=3000 then concave down

   $\left.\begin{array}{l} P=4000 \\ P=8000 \end{array}\right\}$ Concave down

   (d) Use Euler's Method with step size $h = 1$ to estimate the population after 50 years if the initial population is 1000.

   $P(50) \approx 1064.04$

   (e) If the initial population is 1000, write a formula for the population after t years. Use it to find the population after 50 years and compare with your estimate in part (d).

   $$P(t) = \frac{6000}{1 + 5e^{-0.0015(t)}}$$

   $P(50) \approx 1064.07$

   (f) Graph the solution in part (e) and compare with the solution curve you sketched in part (c).

   identical

3. Make a guess as to the carrying capacity for the United States.

   (a) Use your guess and the fact that the population was 250 million in 1990 to formulate a logistic model for the U.S. population.

   $K = 700 \Rightarrow \frac{dP}{dt} = kP\left(1 - \frac{P}{700}\right)$

   (b) Determine the value of k in your model by using the fact that the population in 2000 was 275 million.

   $k = 0.0144$

   (c) Use your model to predict the U.S. population in the years 2100 and 2200.

   $P(110) = 513.49\text{M}$

   $P(210) = 645.29\text{M}$

   (d) Use your model to predict the year in which the U.S. population will exceed 300 million.

   $t \approx 20.6 \Rightarrow 2010$

4. One model for the spread of a rumor is the rate of spread is proportional to the product of the fraction y of the population who have heard the rumor and the fraction who have not heard the rumor.

   (a) Write a differential equation that is satisfied by y.

   $\frac{dy}{dt} = ky(1-y)$

   (b) Solve the differential equation.

   $y(t) = \frac{e^{kt}}{e^{kt} - e^{C}}$

   (c) A small town has 1000 inhabitants. At 8 a.m., 80 people have heard a rumor. By noon, half the town has heard it. At what time will 90% of the population have heard the rumor?

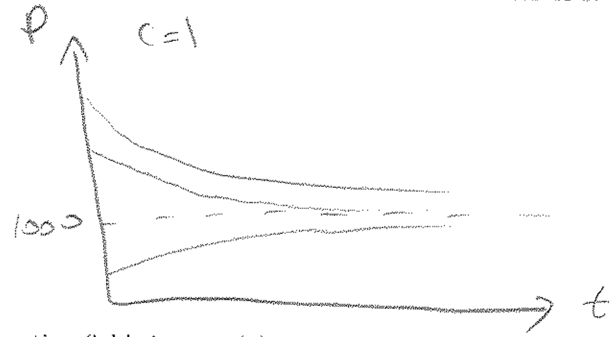
   $t \approx$ 3:36 p.m

5. Consider the differential equation

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - c$$

as a model for a fish population in a fish hatchery, where t is measured in weeks and c is a constant.

(a) Use Mathematica to draw direction fields for various values of c.



(b) From your direction fields in part (a), determine the values of c for which there is at least one equilibrium solution. For what values of c does the fish population always die out?

1 equil. solution $\Rightarrow$ $0 \le c \le 20$

fish pop die $\Rightarrow$ $c > 20$

(c) Use the differential equation to prove what you discovered graphically in part (b).

$$P = \frac{-.08 \pm \sqrt{(.08)^2 - 4(-.00008)(-c)}}{2(-.00008)}$$

when discriminant is nonnegative, there is at least one equil. solution

for $c > 20$, $\frac{dP}{dt} < 0$ and population dies out.

(d) What would you recommend for a limit to the weekly catch of fish population? (What value of C would you recommend?)

less than 20

6. There is considerable evidence to support the theory that for some species there is a minimum population m such that the species will become extinct if the size of the population falls below m. This condition can be incorporated into the logistic equation by introducing the factor $(1 - m/P)$. Thus, the modified logistic model is given by the differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)\left(1 - \frac{m}{P}\right)$$

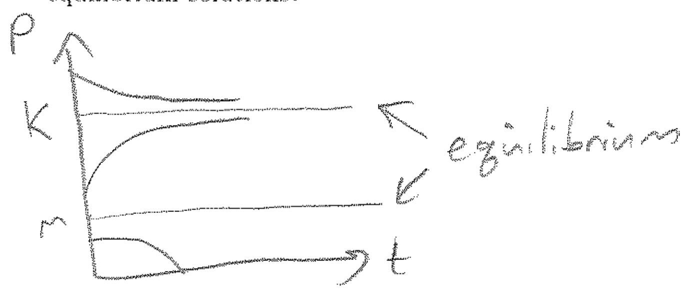
(a) Use the differential equation to show that any solution is increasing if $m < P < K$ and decreasing if $0 < P < m$.

if $P > m$ then $1 - \frac{m}{P} > 0$ and

if $P < K$ then $1 - \frac{P}{K} > 0$ then $\frac{dP}{dt}$ is always > 0

if $m > P > 0$ then $1 - \frac{m}{P} < 0$; $1 - \frac{P}{K} > 0$, $P > 0$ $\therefore$ $\frac{dP}{dt} < 0$

(b) For the case where $k = 0.08$, $K = 1000$, and $m = 200$, draw a direction field and use it to sketch several solution curves. Describe what happens to the population for various initial populations. What are the equilibrium solutions?



(c) Solve the differential equation explicitly, either by using partial fractions or with Mathematica. Use an arbitrary initial condition, $P(0) = P_0$.

$$P(t) = \frac{m(K - P_0) + K(P_0 - m)e^{(K-m)(k/K)t}}{K - P_0 + (P_0 - m)e^{(K-m)(k/K)t}}$$

(d) Use the solution in part (c) to show that if $P_0 < m$, then the species will become extinct. *Hint*: Show that the numerator in your expression for $P(t)$ is 0 for some value of t.

If $P_0 < m$, then $P_0 - m < 0$. Let $N(t)$ be the number of expression for $P(t)$ in part c.

Then $N(0) = P_0(K - m) > 0$ and $P_0 - m < 0$.

$\lim_{t \to \infty} K(P_0 - m)e^{(K-m)(k/K)t} = -\infty$ $\quad \lim_{t \to \infty} N(t) = -\infty$

Since N is continuous, there is a # t such that $N(t) = 0$ and thus $P(t) = 0$ so species will become extinct

7. Biologists stocked a lake with 400 fish and estimated the carrying capacity (the maximum population for the fish of that species in that lake) to be 10,000. The number of fish tripled the first year.

   (a) Assuming that the size of the fish population satisfies the logistic equation, find an expression for the size of the population after t years.

$$P(t) = \frac{10000e^{1.1852t}}{24+e^{1.1852t}}$$

   (b) How long will it take for the population to increase to 5000?

$P(t) = 5000 \Rightarrow t = 2.68145$ years

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