Mechanics Based Problems

1. Write the linear systems in matrix form:

(a) $\dfrac{dx}{dt} = x + 2y$

$\dfrac{dy}{dt} = 4x + 3y$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(b) $\dfrac{dx}{dt} = 3x - y$

$\dfrac{dy}{dt} = 9x - 3y$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & 1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

(c) $\dfrac{dx}{dt} = 6x - y$

$\dfrac{dy}{dt} = 5x + 2y$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 6 & -1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2. Write the given system without the use of matrices:

(a) $\mathbf{X}' = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix} \mathbf{X} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$

$$\frac{dx}{dt} = 4x + 2y + e^t$$

$$\frac{dy}{dt} = -x + 3y - e^t$$

3. Verify that the vector $\mathbf{X}$ is a solution of the given system:

(a) $\dfrac{dx}{dt} = 3x - 4y$

$\dfrac{dy}{dt} = 4x - 7y$

$\mathbf{X} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-5t}$

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$\begin{bmatrix} -5e^{-5t} \\ -10e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{-5t} \\ -10e^{-5t} \end{bmatrix}$ QED

4. Determine whether the vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ are linearly independent:

(a) $\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$, $\mathbf{X}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-6t}$

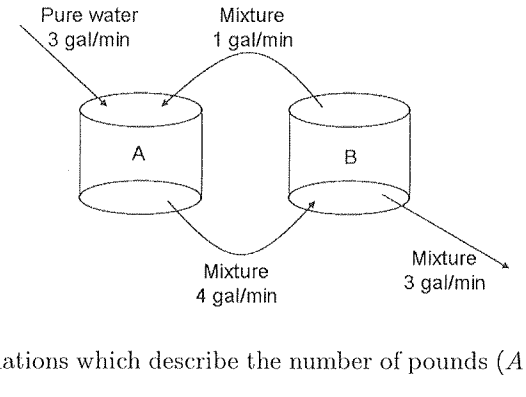
linear independent

(b) $\mathbf{X}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t}$, $\mathbf{X}_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix} e^{t} + \begin{bmatrix} 8 \\ -8 \end{bmatrix} te^{t}$

linear independent

Problem Solving Problems

1. Tank A contains 50 gallons of water in which 25 pounds of salt are dissolved. A second tank, B, contains 50 gallons of pure water. Liquid is pumped in and out of the tanks at rates shown in the figure below.



(a) Derive the differential equations which describe the number of pounds ($A(t)$ and $B(t)$) of salt at any time in tanks A and B.

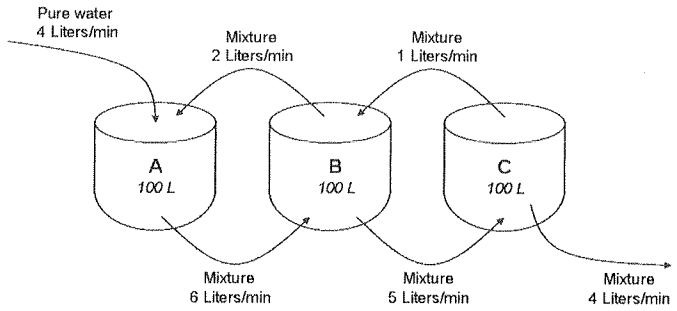
$$\frac{dA}{dt} = \frac{B}{50} - \frac{2A}{25}$$

$$\frac{dB}{dt} = \frac{2A}{25} - \frac{2B}{25}$$

(b) Write this system of differential equations in matrix form.

$$\begin{bmatrix} A \\ B \end{bmatrix}' = \begin{bmatrix} -2/25 & 1/50 \\ 2/25 & -2/25 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}$$

2. Using the information in the figure below, derive the system of differential equations describing the number of mg of salt at any time in tanks A, B, and C, respectively.



$$\frac{dA}{dt} = 0 + \frac{B}{50} - \frac{3A}{50}$$

$$\frac{dB}{dt} = \frac{3A}{50} + \frac{C}{100} - \frac{7B}{100}$$

$$\frac{dC}{dt} = \frac{B}{20} - \frac{C}{20}$$