MA 371 Homework 1. Due the week of 9/8/03

1. (a) Let $\mathbf{v} = [1, 2]$ Find a non-zero vector in $\mathcal{R}^2$ which is orthogonal to $\mathbf{v}$.
   (b) Let $\mathbf{v}_1 = [1, -3, 2]$ and $\mathbf{v}_2 = [-1, 1, 1]$ Find a non-zero vector in $\mathcal{R}^3$ orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$.
   (c) Find a non-zero vector in $\mathcal{R}^5$ which is orthogonal to $\mathbf{v}_1 = [0, 2, 1, 0, 2], \mathbf{v}_2 = [1, 0, 0, 2, 0], \mathbf{v}_3 = [0, 1, 2, 0, 1]$, and $\mathbf{v}_4 = [2, 0, 0, 1, 0]$.
   (d) Show that there does not exist a non-zero vector in $\mathcal{R}^5$ which is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, and also to the vector you found in part (c).

2. Suppose $\mathbf{x}$ and $\mathbf{y}$ are vectors such that $\mathbf{x} = k\mathbf{y}$ for some scalar, k. For each of the following statements, either show that the statement is true, or give an example which shows that the statement is false. If the statement is false, give a condition on k which makes the statement true, and show this.

   (a) Equality holds in the Triangle Inequality for the vectors $\mathbf{x}$ and $\mathbf{y}$; that is, $\|\mathbf{x} + \mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$.
   (b) Equality holds in the Cauchy-Schwarz Inequality for the vectors $\mathbf{x}$ and $\mathbf{y}$; that is, $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\|$.

3. (From Section 1.2 Problem D9) For each of the following, either show that the statement is true, or give examples of vectors $\mathbf{u}$ and $\mathbf{v}$ for which the statement is false.

   (a) If $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   (b) If $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}$ is orthogonal to $\mathbf{v} + \mathbf{w}$.
   (c) If $\mathbf{u}$ is orthogonal to $\mathbf{v} + \mathbf{w}$, then $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$.

4. Give examples of vectors $\mathbf{u}$ and $\mathbf{v}$ such that the following hold. Illustrate by drawing a diagram.

   (a) $\|\mathbf{u} + \mathbf{v}\| < \|\mathbf{u}\| + \|\mathbf{v}\|$.
   (b) $\|\mathbf{u} + \mathbf{v}\|^2 < \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.
   (c) $\|\mathbf{u} + \mathbf{v}\|^2 > \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$.

5. (From Sec. 2.2 Problem 47) Consider the systems

$$\begin{aligned} 2x + 3y - z &= 1 \\ 4x + 6y - 2z &= 0 \end{aligned}, \text{ and } \begin{aligned} 2x + 3y - z &= 1 \\ 4x + y - 2z &= 0 \end{aligned}.$$

   For each system, what does the nature of solutions (i.e., unique, infinite, no solution) tell you about

   (a) The planes represented by the equations?
   (b) The vectors in $\mathcal{R}^2$ represented by the columns of the coefficient matrix?

6. Let $A = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 1 & -1 \\ 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 3 & -2 & 2 \end{pmatrix}$, and $\mathbf{b} = [4, -3, -1, 0, 1]^T$.

(a) Find the reduced row echelon form of A.

(b) Solve the system $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x}$.

(c) Solve the system $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$.

7. Let $A = \begin{pmatrix} 0 & k & 1 & 0 & k \\ 1 & 0 & 0 & k & 0 \\ 0 & 1 & k & 0 & 1 \\ k & 0 & 0 & 1 & 0 \\ 0 & k & 1 & 0 & 2k \end{pmatrix}$, and $\mathbf{b} = [0, 1, 2, 2, 3]^T$.

(a) Find all values of k such that $A\mathbf{x} = \mathbf{b}$ has a unique solution. Find this solution.

(b) For the values of k for which a unique solution does not exist, use the reduce row echelon form of the augmented matrix $A|\mathbf{b}$ to show either that infinite solutions exist or that no solutions exist.

(c) Suppose that k is such that $A\mathbf{x} = \mathbf{b}$ has no solution. Does there exist a non-zero vector $\mathbf{c}$ such that $A\mathbf{x} = \mathbf{c}$ does have a solution? If so, find one. Is it unique?