MA 371 Homework 2 solutions

1.

$$\begin{aligned}(A^{-1}B^TC)^{-1}[CB^{-1}(A^{-1})^T]^T &= C^{-1}(B^T)^{-1}AA^{-1}(B^{-1})^TC^T \\ &= C^{-1}(B^T)^{-1}I(B^T)^{-1}C^T \\ &= C^{-1}(B^T)^{-2}C^T.\end{aligned}$$

2. Sec. 3.2, #33.

If A, B, and $A+B$ are invertible, then

$$\begin{aligned}A(A^{-1}+B^{-1})B(A+B)^{-1} &= (AA^{-1}+AB^{-1})B(A+B)^{-1} \\ &= (IB+AB^{-1}B)(A+B)^{-1} \\ &= (B+A)(A+B)^{-1} \\ &= (A+B)(A+B)^{-1} \\ &= I.\end{aligned}$$

It follows that $A^{-1}+B^{-1}$ is invertible. In fact, since $A(A^{-1}+B^{-1})B(A+B)^{-1}=I$, we have

$$\begin{aligned}(A^{-1}+B^{-1})B(A+B)^{-1} &= A^{-1} \\ &\Longleftrightarrow \\ (A^{-1}+B^{-1})B(A+B)^{-1}A &= A^{-1}A=I,\end{aligned}$$

so $(A^{-1}+B^{-1})^{-1}=B(A+B)^{-1}A$.

3. If $A=\begin{pmatrix} k & 1 & 0 & 1 \\ 1 & k & 1 & 0 \\ 0 & 1 & k & 1 \\ 1 & 0 & 1 & k \end{pmatrix}$,

$$A^{-1}=\frac{1}{-4k^2+k^4}\begin{pmatrix} -2k+k^3 & -k^2 & 2k & -k^2 \\ -k^2 & -2k+k^3 & -k^2 & 2k \\ 2k & -k^2 & -2k+k^3 & -k^2 \\ -k^2 & 2k & -k^2 & -2k+k^3 \end{pmatrix},$$

provided that $det(A)=-4k^2+k^4\neq 0$, which is true if $k=0, k=2$, or $k=-2$. Notice that A^{-1} is symmetric, as is A.

4. Let $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 \\ c & b & a & 0 & 0 \\ d & c & b & a & 0 \end{pmatrix}$. Then $A^n = \mathbf{0}$ for $n \geq 5$. Note that A is lower triangular with zeros on the diagonal. What do you think happens if A is *upper* triangular with zeros on the diagonal?

5. (a) If $E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ is such that the product EA has the effect of interchanging rows 2 and 4 of an arbitrary $4 \times n$ matrix A.

   (b) If $E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -5 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ is such that the product EA has the effect of multiplying row 3 of an arbitrary $4 \times n$ matrix A by -5 and adding it to row 2.

6. The desired matrix is $E_1E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -5 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$.

7. Section 3.4 # 20. If the vectors *are* dependent, express v_1 as a linear combination of the other vectors.

   (a) We find $RREF \begin{pmatrix} 3 & 1 & 2 & 1 \\ 8 & 5 & -1 & 4 \\ 7 & 3 & 2 & 0 \\ -3 & -1 & 6 & 3 \end{pmatrix} = I_{4\times 4}$, so the vectors are independent.

   (d) Let $A = \begin{pmatrix} 4 & 9 & 4 & 6 & 4 \\ 7 & 7 & -3 & 4 & -7 \\ 6 & 2 & 1 & 9 & 0 \\ -2 & -1 & 5 & 1 & 6 \end{pmatrix}$. The $RREF(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{508}{249} \\ 0 & 1 & 0 & 0 & \frac{106}{249} \\ 0 & 0 & 1 & 0 & \frac{55}{249} \\ 0 & 0 & 0 & 1 & \frac{103}{249} \end{pmatrix}$, so the vectors $v_1, v_2, v_3, v_4, and v_5$ are dependent. In fact, $Ax = 0$ has solution

$$x_4 = -\frac{103}{249}x_5$$
$$x_3 = -\frac{55}{249}x_5$$
$$x_2 = -\frac{106}{249}x_5$$

$$x_1 = \frac{508}{249} x_5,$$

so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_5 \begin{pmatrix} \frac{508}{249} \\ -\frac{106}{249} \\ -\frac{55}{249} \\ -\frac{103}{249} \\ 1 \end{pmatrix}.$$

So, for example, if $x_5 = 249$, (to get rid of the fractions) we have

$$508v_1 - 106v_2 - 55v_3 - 103v_4 + v_5 = 0.$$

This explicitly shows linear dependence. Also, we have

$$v_1 = \frac{1}{508}(106v_2 + 55v_3 + 103v_4 - v_5).$$

8. Section 3.4 # 26.

   Any vector of the form $\begin{pmatrix} a \\ b \\ 2a \\ 3b \\ -a \end{pmatrix}$ where a and b are any real numbers can be written in the form $a \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$. This is the set of all linear combinations, or the *span* of the vectors $\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 0 \end{pmatrix}$, which, by Theorem 3.4.2, is a subspace of $\mathcal{R}^5$.

9. We need to find all vectors x such that $v_1 \cdot x = v_2 \cdot x = 0$. Thus we need to solve $Ax = 0$ where $\begin{pmatrix} 1 & -1 & 0 & -2 \\ -1 & 2 & 1 & 1 \end{pmatrix}$. The $RREF(A) = \begin{pmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & 1 & -1 \end{pmatrix}$, and we find that the solution is *span* $\left(\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right)$.

10. Section 3.5 # 20.
The solution set to the system

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + 4x_4 - x_5 &= 0 \\ 3x_1 + 2x_2 + x_3 + 2x_4 - 3x_5 &= 0 \\ x_1 + 5x_2 \qquad - 2x_4 - 4x_5 &= 0 \end{aligned}$$

consists of all vectors orthogonal to v_1, v_2, and v_3 (and therefore to $span(v_1, v_2, v_3)$ (Why?)). You really need to solve the system, or at least row reduce the associated matrix, (c) to answer (b). I leave this to the reader. **Remarks:** If the vectors are independent (they are), you will get two free variables (Why?) so the solution space will be the span of two independent vectors, which is a plane through the origin in $\mathbf{R}^5$, which is "perpendicular" to the span of the vectors v_1, v_2, and v_3, which is a 3 dimensional subspace of $\mathbf{R}^5$.