MA 371 9/30/03. Solutions

1. Let $A = \begin{pmatrix} 2 & 1 & 3 & -2 \\ 4 & 2 & 1 & -2 \\ 6 & 3 & 4 & -4 \\ 4 & 2 & 1 & -2 \end{pmatrix}$ and $b = (7, 1, 8, 1)$.

   (a) To solve $Ax = 0$, as usual, find

$$RREF(A) = \begin{pmatrix} 1 & 1/2 & 0 & -2/5 \\ 0 & 0 & 1 & -2/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Expressing basic in terms of free variables, we get

$$\begin{aligned} x_3 &= \frac{2}{5}x_4 \\ x_1 &= -\frac{1}{2}x_2 + \frac{2}{5}x_4. \end{aligned}$$

So

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_2 + \frac{2}{5}x_4 \\ x_2 \\ \frac{2}{5}x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{2}{5} \\ 0 \\ \frac{2}{5} \\ 1 \end{pmatrix},$$

where x_1 and x_2 are any real numbers. For example, $v_1 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 5 \end{pmatrix}$ are both solutions. In fact, these vectors are linearly independent and span the solution space for $Ax = 0$. (This space is called the *null space* of A.) Thus the solution space is $span(v_1, v_2)$. Note that we could have taken *any* multiples of v_1 and v_2 integers just look cleaner than fractions.

(b) I'll leave it as an exercise to show directly (i.e., by multiplying by A) that v_1, v_2, and $v_1 + v_2$ are solutions. More interestingly, it is also easy to see that *any* vector of the form $rv_1 + sv_2$, where r and s are scalars, is a solution to $Ax = 0$. Since v_1 and v_2 are both solutions, we have

$$\begin{aligned} A(rv_1 + sv_2) &= A(rv_1) + A(sv_2) \\ &= rAv_1 + sAv_2 \\ &= r \cdot 0 + s \cdot 0 \\ &= 0. \end{aligned}$$

The point is that the solution space, or null space, is a subspace of $\mathbf{R}^4$.

(c) ,

To solve $Ax = b$, find

$$RREF(A|b) = RREF \begin{pmatrix} 2 & 1 & 3 & -2 & 7 \\ 4 & 2 & 1 & -2 & 1 \\ 6 & 3 & 4 & -4 & 8 \\ 4 & 2 & 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 & 0 & -2/5 & -2/5 \\ 0 & 0 & 1 & -2/5 & 13/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So the solution is obtained by adding the vector $\begin{pmatrix} -2/5 \\ 13/5 \\ 0 \\ 0 \end{pmatrix}$ to the homogeneous solution above. That is, the solution is

$$\begin{pmatrix} -2/5 \\ 13/5 \\ 0 \\ 0 \end{pmatrix} + rv_1 + sv_2,$$

where r and s are any real numbers.

(d) Note that the first vector above, which is the non-homogeneous solution, is not multiplied by an arbitrary constant. Thus, for example, $\begin{pmatrix} -2 \\ 13 \\ 0 \\ 0 \end{pmatrix}$ is not a solution. Check this by substitution. Then check, that, for example,

$$v_1 + \begin{pmatrix} -2 \\ 13 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 15 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad v_2 + \begin{pmatrix} -2 \\ 13 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 13 \\ 2 \\ 5 \end{pmatrix}$$

are two solutions (independent) solutions, but their sum, $(-3, 28, 2, 5)$ is not. The point is that the set of solutions to $Ax = b$ is *not* a subspace of $\mathbf{R}^4$.

(e) Row and column space, and bases, will be covered in Ch. 7. But for the record, the two non-zero rows of RREF(A) span the *row space* of A, It's a subspace of $\mathbf{R}^4$. (That's why it's called a space.) Find a basis for the row space of A.

2. See also homework 2 solutions, #7.

To show linear independence, we can find the RREF of the matrix whose **rows** are the given vectors. If we get a row of zeros, the vectors are **dependent**. If we do not get a row of zeros, the vectors are **independent**. **Or** we can find the RREF of the matrix whose **columns** are the given vectors. If we get a free variable the vectors linearly **dependent**. If we do not get free variables, the vectors are **independent**. **However**, only the second method will allow us to express one vector in terms of the others. See also homework 2 solutions, #7. Note that the analysis has nothing to do with whether or not the matrices are square. (If the matrix **is** square, however, then the rows are independent if and only if the columns are independent, which can happen if and only if RREF(A)=I.)

(a) $v_1 = (1, -2, 0), v_2 = (3, -1, 1), v_3 = (1, 2, 3)$ RREF(A)=I. The vectors are independent.

(b) Setting the vectors as columns of a matrix we find$RREF \begin{pmatrix} 0 & 1 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 2 \end{pmatrix} =$ $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, so the vectors are independent and we find the solution to $Ax = 0$, where A is the above matrix is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Since v_1, V_2, v_3 are *columns* of A,

$$0 = Ax = A \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = -1v_1 - 1v_2 + v_3.$$

Solving for v_1 yields $v_1 = v_2 - v_3$.

(c) $v_1 = (1, -1, 0, -2), v_2 = (-1, 2, 1, 1), v_3 = (1, 1, 2, 4)$.

3. For similar problem (partial) solution see Homework 2 solutions #9. Recall that the *orthogonal complement* of a subspace W is space consisting of all vectors orthogonal to every vector in W. (The problem should actually read "Find the orthogonal complement of $span(v_1, v_2)$.) The orthogonal complement is just the solution space to $Ax = 0$ where the given vectors are the **rows** of A. See Theorem 3.5.6.

   For this example, we need to find the solution space to $\begin{pmatrix} 1 & -1 & 0 & -2 \\ -1 & 2 & 1 & 1 \end{pmatrix} x = 0.$

   So find RREF(A), express basic in terms of free, etc, etc. We find $span \left(\begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right)$

   is the orthogonal complement of v_1, v_2.