MA 371 9/12/03. Linear Equations, Matrix Operations Selected Solutions

2. (a) Let $A = \begin{pmatrix} 2 & 1 & 3 & -2 \\ 4 & 2 & 1 & -2 \\ 6 & 3 & 4 & -4 \\ 4 & 2 & 1 & -2 \end{pmatrix}$, and $\mathbf{b} = (2, 1, 8, -4)^T$, The RREF of A is $\begin{pmatrix} 1 & 1/2 & 0 & -2/5 \\ 0 & 0 & 1 & -2/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The basic variables are x_1, and x_3; the free variables are x_2 and x_4. So we see that the solution to $A\mathbf{x} = \mathbf{0}$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}x_2 + \frac{2}{5}x_4 \\ x_2 \\ \frac{2}{5}x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} \frac{2}{5} \\ 0 \\ \frac{2}{5} \\ 1 \end{pmatrix},$$

which can be expressed

$$r \begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 0 \\ 2 \\ 5 \end{pmatrix},$$

where r and s are any two real numbers.

2. (b) To see that $\mathbf{x} = [0, 4, 2, 5]^T$ is indeed a solution to $A\mathbf{x} = \mathbf{0}$, we can check that $A[0, 4, 2, 5]^T = \mathbf{0}$. To see that this agrees with the solution found above, note that $[0, 4, 2, 5]^T$ is the linear combination of the two vectors above when $r = 2$ and $s = 1$.

5. We wish to find a polynomial $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ such that $p(-1) = 1, p(1) = 3, p(2) = -1$, and $p(4) = 10$. We need to find the coefficients, a_0, a_1, a_2, and a_3. We obtain the system of equations

$$\begin{aligned} a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + 1a_0 &= 1 \\ a_3(1)^3 + a_2(1)^2 + a_1(1) + 1a_0 &= 3 \\ a_3(2)^3 + a_2(2)^2 + a_1(2) + 1a_0 &= -1 \\ a_3(4)^3 + a_2(4)^2 + a_1(4) + 1a_0 &= 10. \end{aligned}$$

The system can be written in matrix form as

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 64 & 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ -1 \\ 10 \end{pmatrix}.$$

The system is easily solved by finding the RREF of the above matrix. This is left as an exercise.