Broadcast Covers in Graphs

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Abstract

A *broadcast cover* is a integer valued function f on the vertices of a graph such that every edge uw is distance at most $f(v)$ from some vertex $v \in V$. We can regard the vertices v with $f(v) > 0$ as broadcast stations, each having a transmission power that might be different from the powers of other stations. The optimal broadcast cover problem seeks a broadcast cover that minimizes the sum of the costs of the broadcasts assigned to the vertices of the graph. We present a theorem about the nature of broadcast covers that establishes a polynomial time algorithm for the problem on arbitrary graphs. We also discuss the broadcast domination problem and some interesting relationships between it and broadcast cover.

1 Introduction and motivation

A *vertex cover* S is a subset of the vertices in a graph such that every edge in the graph has at least one endpoint in S. The topic has long been of interest to researchers. The associated decision problem, VERTEX COVER, occupies a prominent place in the computational complexity literature [5]. This definition also leads naturally to the associated optimization problem which is to find a vertex cover of minimum cardinality. Numerous variants of this problem have been studied.

VERTEX COVER is NP-complete on arbitrary graphs [5]. It is also NP-complete on several classes of graphs, including cubic planar graphs and triangle-free graphs. The problem can be solved in polynomial time on, for example, bipartite graphs and chordal graphs [5], as well as recursively constructed graphs such as trees and series-parallel graphs [2, 8].

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We start with notation and more formal problem definitions. All graphs in this paper are simple and connected. Let $G = (V, E)$ be a graph with $n = |V|$ and $m = |E|$. For any vertex $v \in V$, the *neighborhood* of v is the set $N(v) = \{u \mid uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. The *edge neighborhood* of v is the set $N_e(v) = \{uv \mid u \in V\}$. Similarly, for any set $S \subseteq V$, $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = N(S) \cup S$. For any set $S \subseteq V$, $N_e(S) = \cup_{v \in S} N_e(v)$. A set S is a *dominating set* if $N[S] = V$. The minimum cardinality of a dominating set of G is denoted by $\gamma(G)$. A set S is a *vertex cover* if $N_e(S) = E$. The minimum cardinality of a vertex cover of G is denoted by $\beta(G)$. The *distance*, $d(u, v)$, between two vertices u and v in G is the number of edges on a shortest path between u and v in G. The *eccentricity*, $e(v)$, of a vertex v is the largest distance from v to any vertex of G. The *radius* of G, $rad(G)$, is the smallest eccentricity in G. The *diameter* of G, $diam(G)$, is the largest eccentricity in G.

In [4], Erwin introduced a variant of the dominating set problem known as *broadcast domination*. In this version, the vertices in S (known as *dominators*) each have an associated positive integer that can be thought of as its *broadcast range*. Then vertices not in S are dominated if they are within the broadcast range of some dominator. More formally, a *dominating broadcast* is an integer valued function f on the vertices such that every vertex of the graph is distance at most $f(v)$ from some vertex v that has $f(v) > 0$. A dominating broadcast is optimal if it minimizes the sum of the costs of the dominators across all vertices in the graph. These costs are typically taken to be the $f(v)$ values. Some related problems are discussed in [3].

In this paper, we introduce the broadcast cover problem. Loosely speaking, broadcast cover is to vertex cover what broadcast domination is to dominating set. As with broadcast domination, the vertices in S (known as covering vertices) provide *coverage* to the edges of the graph. A formal definition appears in Section 4.

This paper is organized in the following way. The next section formalizes the notion of broadcasts in graphs, providing background material common to both broadcast domination and broadcast cover. Following, we provide a brief summary of results for the broadcast domination problem. In Section 4, we formally define the broadcast cover problem and provide some new results. Section 5 contains a comparison of the two problems and some concluding remarks.

2 Broadcasts in Graphs

Given a graph $G = (V, E)$, a *broadcast* is a function $f : V \to \{0, 1, 2, ...\}$. The *cost* of a broadcast f incurred by a set $S \subseteq V$ is $f(S) = \sum_{v \in S} f(v)$. Thus, $f(V)$ is the total cost incurred by the broadcast function f. More informally, we say that vertex v has an *originating broadcast of power* $f(v)$ if $f(v) > 0$. If $f(v) = 0$ we say v has no originating broadcast. If $f(v) > 0$ and $d(u, v) < f(v)$, we say the broadcast originating from vertex v reaches vertex u with power $f(v) - d(u, v)$.

A broadcast is *dominating* if for every vertex $u \in V$, there is a vertex v with $f(v) > 0$ and $d(u, v) \leq f(v)$. A dominating broadcast f is *optimal* if $f(V)$ is minimum over all choices of broadcast dominating functions for G. An optimal broadcast dominating function satisfies $f(V) \leq rad(G)$, since if v is a vertex of minimum eccentricity (*i.e.* a *center* of the graph), the broadcast function defined by $f(v) \leftarrow rad(G)$ and $f(u) \leftarrow 0$ for all $u \neq v$ is dominating. We denote the cost of an optimal broadcast dominating function as $\gamma_b(G)$. We will call any broadcast dominating function with only one non-zero $f(v)$ value a γ_b-radial broadcast, and we call a *graph* γ_b-radial if some γ_b-radial broadcast is optimal.

A broadcast is *covering* if for every edge $uv \in E$, there is a path in G that satisfies two properties. First, the path must include the edge uv. Second, one end of the path must be a vertex w with $f(w)$ at least the length of the path. This second condition could be restated to say there must exist a $w \in V$ with $f(w) \geq \max\{d(w, v), d(w, u)\}$ where the distances are measured in the specified path. Informally this means that every edge uv in the graph is within distance $f(w)$ from some vertex w, where the distance from uv to w is defined as the number of vertices between uv and w inclusive of w and one of $\{u, v\}$. A covering broadcast f is *optimal* if $f(V)$ is minimum over all choices of broadcast covering functions for G. We denote the cost of an optimal broadcast covering function as $\beta_b(G)$. We will call any broadcast covering function with only one non-zero $f(v)$ value an β_b-radial broadcast, and we call a *graph* β_b-radial if some β_b-radial broadcast is optimal.

3 Broadcast Domination

Since its introduction by Erwin [4], the broadcast domination problem has been studied by several authors. If the problem is modified to include a fixed maximum allowable broadcast power as a part of the instance it is easy to see, by setting that value to 1, that the resulting problem is NP-hard since it reduces to DOMINATING SET. The broadcast domination problem is solvable in polynomial time on paths, cycles, complete graphs, and grid

graphs [3]. Polynomial time algorithms for the problem on interval graphs, series-parallel graphs, and trees are given in [1]. The algorithms for trees and series-parallel graphs rely on theory concerning recursive graph algorithms developed in [8] and [2]. These algorithms solve instances of the variant with a fixed maximum allowable broadcast power as well as the general case. In [7], the authors formulate the problem as an integer program and report on computational results. Quite recently, Heggerness and Lokshtanov [6] have established that broadcast domination (with no fixed maximum broadcast power) can be solved in polynomial time on arbitrary graphs.

A dominating broadcast is *efficient* if for every $v \in V$, there is *exactly* one $u \in V$ satisfying $d(u, v) \leq f(u)$. In [3], the authors prove that every graph has an efficient broadcast f satisfying $f(V) = \gamma_b(G)$. The proof of this theorem is constructive in the sense that it shows how any broadcast can be transformed into an efficient broadcast with the same total cost. This result turns out to be essential to our results in the next section.

Recall the concept of γ_b-radial broadcasts and γ_b-radial graphs from the previous section. By considering a simple path on 6 vertices $v_1, v_2, ..., v_6$ we can observe that not all graphs are γ_b-radial. The minimum γ_b-radial broadcast has $f(V) = 3$, but the broadcast f' with $f'(v_2) = f'(v_5) = 1$ achieves $f'(V) = 2$. Now we turn our attention to the analogous question for broadcast cover. That is, are all connected graphs β_b-radial? The answer, somewhat surprisingly, turns out to be yes.

4 Broadcast Cover

The main result of this section is a proof that all connected graphs are β_b-radial. We start with some simple lemmas.

Lemma 4.1 *Every optimal broadcast covering function satisfies* $f(V) \leq rad(G) + 1$.

Proof. Consider a center vertex v and define the broadcast function $f(v) \leftarrow rad(G)$ and $f(u) \leftarrow 0$ for all $u \neq v$. Either f is covering or it is not. If it is, $f(V) \leq rad(G)$ and we are done. If f is not covering, observe that f nevertheless reaches every vertex of G (*i.e.*, f is broadcast dominating). Only edges with both endpoints at distance $rad(G)$ from v are not covered. Modify f by increasing $f(v)$ by 1 and the result is broadcast covering for G. ∎

Lemma 4.2 *If G is β_b-radial, then $\beta_b(G)$ is either $rad(G)$ or $rad(G)+1$.*

Proof. By the previous lemma, $\beta_b(G) \leq rad(G)+1$ holds for all graphs. If G is β_b-radial, then there must exist an optimal broadcast covering function f with only one non-zero $f(v)$ value. But that value must be at least $rad(G)$ or it cannot cover every edge in G. ∎

Both possible $\beta_b(G)$ values can be realized by β_b-radial graphs. For example, if G is a star graph, assigning $f(v) \leftarrow rad(G)$ to the center vertex v is enough to cover the graph. But for the wheel graph $f(v) \leftarrow rad(G)+1$ is required.

4.1 Finding Optimal Broadcast Covering Functions

Now we consider the issue of finding optimal broadcast covering functions. We begin with a definition.

Definition 4.3 *Given a graph G and a broadcast covering function f, a* **dominated path** *is a path $u_0, u_1, ..., u_j$ in G with $f(u_0) > 0, f(u_j) > 0$, and some u_i, $0 \leq i \leq j$ satisfying $d(u_0, u_i) \leq f(u_0)$ and $d(u_i, u_j) \leq f(u_j)$, where the distances are measured in the specified path.*

Lemma 4.4 *If f is broadcast covering for G and f is not β_b-radial, then a dominated path exists in G.*

Proof. Consider a broadcast covering function f that is not radial. Accordingly, there must exist distinct vertices u and v with $f(u) > 0$ and $f(v) > 0$. Since G is connected, consider a shortest uv path. If $d(u,v) \leq f(u) + f(v)$ then this path is a dominated path. Otherwise $d(u,v) > f(u) + f(v)$. Consider the vertices u_i and u_{i+1} on the path with $d(u, u_i) = f(u)$ and $d(u, u_{i+1}) = f(u) + 1$. The edge $u_i u_{i+1}$ is not covered by broadcasts originating from u or v, but since f is broadcast covering there must be some w distinct from u with $d(u_i, w) \leq f(w)$ and $d(u_{i+1}, w) \leq f(w)$. The original path from u to u_i, together with a shortest path from u_i to w is then a dominated path. ∎

Now we establish the main result of this section which is that broadcasts that are not β_b-radial can be transformed into β_b-radial broadcasts essentially by repeatedly combining originating broadcasts until only one remains.

Theorem 4.5 *Every graph G is β_b-radial.*

Proof. Consider an optimal broadcast covering function f. If f is not radial, let u_0 and u_j be the endpoints of a dominated path in G with respect to f. Then we create a new function f^* as follows. $f^*(u_{f(u_j)}) \leftarrow f(u_0) +$

$f(u_j)$, $f^*(u_0) \leftarrow 0$, $f^*(u_j) \leftarrow 0$, and $f^*(v) \leftarrow f(v)$ for $v \notin \{u_{f(u_j)}, u_0, u_j\}$. The new function f^* is broadcast covering since the broadcast originating at $u_{f(u_j)}$ reaches u_0 with power $f(u_0)$ and u_j with power $f(u_j)$. The new function f^* satisfies $f^*(V) = f(V) = \beta_b(G)$ by construction. Now set $f(v) \leftarrow f^*(v)$ and repeat the procedure until no path of the specified type exists. Then by Lemma 4.4 the result must be β_b-radial. ∎

Thus any optimal broadcast can be transformed into an optimal β_b-radial broadcast by the procedure outlined in the proof above. Note that the optimality condition is not necessary; any covering broadcast that is not β_b-radial can be transformed into a β_b-radial broadcast of the same total cost.

This result leads immediately to a polynomial time algorithm for computing $\beta_b(G)$ for any graph. Since every graph admits a radial broadcast that is optimal, one can simply consider each vertex in turn as a candidate for the originating broadcast. At each vertex v, it must also be determined if assigning $f(v) \leftarrow e(v)$ covers the graph, or if $e(v) + 1$ is required instead, but this computation is straightforward and efficient.

In light of the previous result, it is interesting to observe that the following more restricted problem is NP-complete.

> RESTRICTED BROADCAST COVER: Given a graph $G = (V, E)$, and positive integers $K, M \leq |V|$, is there a broadcast cover f of G such that $f(V) = \sum_{v \in V} f(v) \leq K$ and $\max_{v \in V} f(v) \leq M$?

If we set $M = 1$, this problem reduces immediately to VERTEX COVER.

It is a straightforward exercise to modify the recursive algorithms from [1] to efficiently solve RESTRICTED BROADCAST COVER.

5 Conclusion

This paper introduced the broadcast cover problem and identified some relationships between it and the broadcast domination problem. The problems are similar in that they can both be solved in polynomial time on arbitrary graphs. They also share the property that the restricted version of the problem is NP-complete. An interesting distinction is the fact that all connected graphs are β_b-radial, while not all are γ_b-radial. Recall from Lemma 4.2 that $rad(G) \leq \beta_b(G) \leq rad(G) + 1$. It is interesting to note that although $\gamma_b(G) \leq rad(G)$, any lower bound on $\gamma_b(G)$ must be arbitrarily distant from $rad(G)$. To see this, consider P_n, the path on n vertices. While $\gamma_b(P_n) = \lceil \frac{n}{3} \rceil$, $rad(P_n) = \lfloor \frac{n}{2} \rfloor$. This difference can be as great as $\frac{n}{6}$.

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