On Some Problems Related to Dominating Set

Steven B. Horton
Department of Mathematical Sciences
United States Military Academy
West Point, New York 10996 USA
steve.horton@usma.edu

Abstract

This paper provides a brief survey of two graph problems related to dominating set: domination equivalence and broadcast domination. The former concerns properties of two disjoint sets of vertices having the same neighborhood. The latter is a generalization of the standard dominating set problem where dominating vertices can "broadcast" that domination to variable distances from their source.

1 Introduction and motivation

A *dominating set* S is a subset of the vertices in a graph such that every vertex in the graph either belongs to S or has a neighbor in S. The topic has long been of interest to researchers [12, 13]. The associated decision problem, DOMINATING SET, occupies a prominent place in the computational complexity literature [11]. This definition also leads naturally to the associated optimization problem which is to find a dominating set of minimum cardinality. Numerous variants of this problem have been studied [1, 12, 13, 19].

DOMINATING SET is NP-complete on arbitrary graphs [11]. It is also NP-complete on several classes of graphs, including planar graphs [11], bipartite graphs [6], and chordal graphs [4]. The problem can be solved in polynomial time on, for example, AT-free graphs [17], permutation graphs [10], interval graphs [9], and trees [11].

We start with notation and more formal problem definitions. All graphs in this paper are simple. Let $G = (V, E)$ be a graph with $n = |V|$ and $m = |E|$. For any vertex $v \in V$, the *neighborhood* of v is the set $N(v) = \{u \mid uv \in E\}$ and the *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. Similarly, for any set $S \subseteq V$, $N(S) = \cup_{v \in S} N(v) - S$ and $N[S] = N(S) \cup S$. A set S is a *dominating set* if $N[S] = V$. The minimum cardinality of a

dominating set of G is denoted by $\gamma(G)$. The *distance*, d_{uv}, between two vertices u and v in G is the smallest number of edges on a path between u and v in G. The *eccentricity*, $e(v)$, of a vertex v is the largest distance from v to any vertex of G. The *radius*, $rad(G)$, of G is the smallest eccentricity in G. The *diameter*, $diam(G)$, of G is the largest eccentricity in G.

The question of when a graph has two disjoint subsets of vertices with the same neighborhoods has recently received attention [14, 16, 2]. Related optimization questions include finding a largest and smallest non-trivial such subset for both open and closed neighborhoods. Collectively, these are known as *domination equivalence* problems [2].

In [8], Erwin introduced another variant of the dominating set problem known as *broadcast domination*. In this version, the vertices in S (known as *dominators*) each have an associated nonnegative integer that can be thought of as *broadcast range*. Then vertices not in S are dominated if they are within the broadcast range of some dominator. More formally, we define a *dominating broadcast* as an integer valued function f on the vertices such that every vertex of the graph is distance at most $f(v)$ from some vertex v that has $f(v) > 0$. A dominating broadcast is optimal if the sum of the costs of the dominators across all vertices in the graph is minimized over all choices of such functions. These costs are typically taken to be the $f(v)$ values. Some related problems are discussed in [7].

This paper is organized in the following way. Some problems related to domination equivalence are discussed in the next section. Section 3 provides an overview of the broadcast domination problem. Section 4 contains a brief summary and some concluding remarks.

2 Domination Equivalence

Given a graph $G = (V, E)$, a DE *pair* (domination-equivalent pair) consists of two disjoint sets R and B with $N[R] = N[B]$. Similarly, an ODE *pair* (open domination-equivalent pair) consists of two disjoint sets R and B with $N(R) = N(B)$. One can think of R and B as red and blue colored vertices, respectively, with the constraint requiring that every vertex in the graph that has a red neighbor also has a blue neighbor, and vice-versa. If red and blue represent competing sides in some contest, then colorings of V that satisfy $N[R] = N[B]$ (or $N(R) = N(B)$) represent cases where, at every $v \in V$, the influence of red and blue is in some sense balanced.

A trivial DE or ODE pair can be obtained in any graph if R and B are empty. In addition, if G has isolated vertices, another trivial ODE pair is obtained by coloring isolates red or blue in an arbitrary fashion. We will ignore these cases.

Necessary and sufficient conditions for the existence of non-trivial DE and ODE pairs are known. In [19], Ore observed that the complement of any minimal dominating set is also dominating. Thus a non-trivial DE pair always exists in a connected graph. Jacobs and Jamison [16] have shown that a non-trivial ODE pair exists if and only if the graph has an adjacency matrix that is not sign-nonsingular. This property can be detected in polynomial time following results of Robertson, Seymour, and Thomas [20] and McCuaig [18]. An interesting corollary is that a tree has an ODE pair if and only if it does not have a perfect matching.

A DE pair is said to be *full* if $N[B] = N[R] = V$. Similarly, an ODE pair is full if $N(B) = N(R) = V$. Full DE pairs exist for all graphs; take any minimal dominating set and its complement. On the other hand, deciding if a graph has a full ODE pair is equivalent to asking if it has two disjoint total dominating sets which was shown NP-complete in [14].

From an optimization standpoint, there are several questions that could be addressed. One could investigate the cardinality of the common neighborhood, the combined cardinality of the red and blue sets, or the cardinality of just one of these sets. This paper will report on the latter, with four natural questions resulting. We use the notation $de(G)$ to represent the minimum cardinality of a member of a non-trivial DE pair, and $DE(G)$ to represent the maximum cardinality of a member of a DE pair. We use $ode(G)$ and $ODE(G)$ analogously for ODE pairs.

First consider $DE(G)$.

Theorem 2.1 (Blair et. al. [2]) *For every graph G without isolates, $DE(G) = n - \gamma(G)$.*

This result follows from the observation that any DE pair can be extended to a full pair by adding uncolored vertices to the red and blue sets. The complexity of computing $DE(G)$ for arbitrary graphs is therefore linked to computing $\gamma(G)$. Similarly, algorithms for computing $\gamma(G)$ for special graphs classes like trees and series-parallel graphs can be used to compute $DE(G)$ as well. There is a large and well-established literature on such algorithms [5, 21].

Blair et. al. [2] provides bounds and complexity results for arbitrary graphs, as well as efficient algorithms for computing $de(G)$, $ode(G)$, and $ODE(G)$ when G is a tree. Their results for bounds are summarized below.

Theorem 2.2 *For any graph G of order n:*
(a) $1 \leq de(G) \leq n/2$ *if G is nonempty.*
(b) $1 \leq ode(G) \leq n/2$ *if G has an* ODE *pair.*
(c) $1 \leq ODE(G) \leq n - 2$ *if G has an* ODE *pair and is isolate free.*
These bounds are all sharp except for the upper bound on $de(G)$.

The work of Blair et. al. [2] also provides algorithms to compute $de(G)$, $ode(G)$, and $ODE(G)$ for trees. They further establish that computing these values for arbitrary graphs is NP-complete. The proofs involve reductions from variants of the SATISFIABILITY problem.

3 Broadcast Domination

All graphs in this section are connected. A function $f : V \to \{0, 1, ..., diam(G)\}$ is a *broadcast* if for every vertex $v \in V$, $f(v) \leq e(v)$. The set of *broadcast dominators* defined by f is the set $V_f = \{u \mid f(u) > 0\}$. The set of vertices that a vertex v can *hear* is $H_f(v) = \{u \in V \mid d_{uv} \leq f(u)\}$. We will omit the subscript f when the broadcast function is clear from the context. The *cost* of a broadcast f incurred by a set $S \subseteq V$ is $f(S) = \sum_{v \in S} f(v)$. Thus, $f(V)$ is the total cost incurred by broadcast function f.

A broadcast is *dominating* if $|H(v)| \geq 1$ for every vertex v of G. The term $\gamma_b(G)$ denotes the minimum cost of a dominating broadcast on G. We will refer to a dominating broadcast of cost $\gamma_b(G)$ as an *optimal broadcast*.

Although the range of a broadcast function is $\{0, 1, ..., diam(G)\}$, we never need to assign values larger than $rad(G)$ to any vertex in order to achieve an optimal broadcast. If we choose a vertex v of minimum eccentricity and assign $f(v) = rad(G)$ while assigning $f(w) = 0$ to all other vertices w, the resulting broadcast is dominating for G. We will call any dominating broadcast with only one non-zero $f(v)$ value a *radial broadcast*. Graphs for which some radial broadcast is an optimal broadcast are called *radial* [7].

A broadcast is *efficient* if every vertex hears exactly one broadcast, that is, for every v, $|H(v)| = 1$. The following interesting result is from [7].

Theorem 3.1 (Dunbar et. al. [7]) *Every graph G has a $\gamma_b(G)$-broadcast that is efficient.*

The proof of this theorem is constructive in the sense that it shows how any broadcast can be transformed into an efficient broadcast with the same total cost.

If we impose a fixed upper bound on $f(v)$, we can prove a complexity result for the problem.

> RESTRICTED BROADCAST DOMINATION: Given a graph $G = (V, E)$, and positive integers $K, M \leq |V|$, is there a broadcast domination f of G such that $\sum_{v \in V} f(v) \leq K$ and $\max_{v \in V} f(v) \leq M$?

Theorem 3.2 RESTRICTED BROADCAST DOMINATION *is NP-complete.*

Proof. If we set $M = 1$, this problem reduces immediately to DOMINATING SET. ∎

Note that the complexity status of broadcast domination (without any broadcast power restrictions) is open.

The standard broadcast domination problem can be solved efficiently for a variety of special graph classes. Solutions for paths, cycles, complete graphs, and grid graphs are given in [7]. In [3], polynomial time algorithms are given for trees, series-parallel graphs, and interval graphs. Crudely, these algorithms recursively decompose a given instance of the problem into smaller instances, solve related problems for the small instances, and then assemble the results to form solutions for the original instance. The algorithms for trees and series-parallel graphs rely on theory concerning recursive graph algorithms developed in [21] and [5]. Some of the algorithms from [3] also solve RESTRICTED BROADCAST DOMINATION for the stated graph classes.

Some experimental work has been done for the broadcast domination problem as well. In [15], the authors formulate the problem using integer programming techniques and then solve a variety of test instances using CPLEX optimization software.

4 Summary

This paper summarizes current research on two problems related to dominating set.

There are several open questions that merit further research. In addition to measuring optimality in different ways as suggested in Section 2, a more general domination eqiuvalence-like problem could be investigated where there are $k > 2$ disjoint sets $S_1, S_2, \ldots, S_k$ with $N[S_1] = N[S_2] = \ldots = N[S_k]$. Another interesting question is the complexity status for the standard broadcast domination problem for arbitrary graphs.

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