A MATH, SCIENCE, AND TECHNOLOGY INTERDISCIPLINARY LIVELY APPLICATION OF MATHEMATICS AT THE UNITED STATES MILITARY ACADEMY

by

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ABSTRACT: This paper reports on an Interdisciplinary Lively Application Project (ILAP) that was administered at the United States Military Academy to assess cadet progress towards the attainment of the Academy's mathematics, science, and technology Academic Program goal. Included is a general outline of the project, an overview of student performance, and some directions for the future.
KEYWORDS: ILAP, Assessment, Project, Mathematics, Statistics, Physics

1 INTRODUCTION

The United States Military Academy (USMA) offers a four year experience that has significant academic, physical, and military components. The academic program culminates in the award of the Bachelor of Science degree for each graduate. The Military Academy's general educational goal is clearly stated in its Concept for Intellectual Development: "To enable its graduates to anticipate and to respond effectively to the uncertainties of a changing technological, social, political, and economic world."

1.1 Mathematics, Science, and Technology at USMA

Under the general education goal, the Dean of the Academic Board has established a set of nine Academic Program goals. To attain these goals, cadets are required to take a large set of "core" courses; 16 in the humanities and social sciences and 15 in mathematics, science, and engineering. The latter consists of 4 mathematics courses, 2 physics courses, 2 chemistry courses, 1 computer science course, 1 terrain analysis course, and 5 engineering science/design courses. All of these except the engineering courses appear in the first four semesters in the typical cadet's academic program of study.

One of the Academic Program goals calls for cadets to "understand and apply the mathematical, physical, and computer sciences to reason scientifically, solve quantitative problems, and use technology." We shall refer to this as the *mathematics, science,* and *technology* goal. It is the *assessment* of this goal that is one of the themes of this paper.

1.2 MA206

The four core mathematics courses at USMA, in the order they appear, are Discrete Dynamical Systems, Calculus I, Calculus II, and Probability and Statistics. MA206, Probability and Statistics, is a 40 semester-hour calculus based introduction to probability and statistics. The course covers descriptive statistics, classical probability, point and interval estimation, hypothesis testing, and an introduction to linear regression. Due primarily to its place at the end of the core mathematics program, MA206 was designated to include a student project that integrated concepts from several mathematics, science, and engineering courses at USMA.

1.3 About this paper

This paper reports on the project the cadets were asked to do that relates to the mathematics, science, and technology goal that is outlined above. In section 2, we generally discuss the contents of this project, called the "Math, Science, and Technology Interdisciplinary Lively Application Project", or MST ILAP. Section 3 contains an after action report that gives some details on how the cadets did on the project, some other pedagogical results, and plans for the future.

2 THE MST ILAP

The MST ILAP was designed to assess the cadets' academic progress upon the completion of four semesters of core mathematics, science, and engineering courses. The project was developed by instructors of MA206 and PH202, the second core physics course. The MST ILAP consisted of four major parts: Data Collection, the RC Circuit, the RLC Circuit, and Statistical Analysis.

2.1 Data Collection

Several weeks before we issued the MST ILAP to the 785 cadets enrolled in MA206, they were organized into groups of three or four. Each group contained at least one cadet who was also enrolled in PH202. A large majority of these 785 cadets were concurrently enrolled in PH202, and most of the remainder had already completed PH202. Only a small minority (less than 5%) had not yet started PH202.

Through prior arrangement with the director of PH202, these cadets collected data that would be used for the MST ILAP during a regularly scheduled physics lab period. The Department of Physics conducted this lab to demonstrate that the capacitance of a series RC circuit could be calculated by measuring the time it took to achieve a certain voltage across the capacitor at specified resistance settings. At one specific resistance setting, the cadets took 50 such time observations. These 50 data points were all observations of the exact same phenomenon, and therefore were only different because of error. This data, known as the "RC Time" data, would be used for the entire project to follow.

2.2 RC Circuit

Next, the cadets were asked to investigate a typical RC circuit analytically. When a potential difference is applied across a capacitor of such a circuit, the rate at which it charges depends on its capacitance, C, and the resistance in the circuit, R. We will use q to represent charge and i to represent current. Both q and i are functions of time, t. Finally, we use ϵ to represent the electromotive force (emf) provided by the battery. A typical RC circuit is shown in Figure 1.

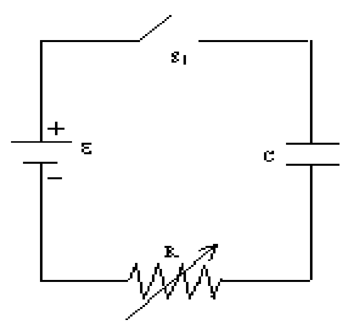

Figure 1: RC Circuit

The cadets were then asked to derive a differential equation for this system using Kirchhoff's Loop Rule. Also known as Kirchhoff's voltage law, this rule states that the algebraic sum of the changes in potential encountered in a complete traversal of any loop of a circuit must be zero. When applied to Figure 1, this yields

$$\epsilon - \frac{q}{C} - iR = 0$$

where ϵ again represents the emf provided by the battery, and q/C and iR are the voltage drops across the capacitor and resistor, respectively. Since $i = \frac{dq}{dt}$, we can rewrite this as the linear, first order, nonhomogeneous differential equation

$$\frac{dq}{dt} + \frac{q}{RC} = \frac{\epsilon}{R}.$$

The cadets were then asked to find the general solution to this differential equation with charge as a function of time. Elementary techniques can be employed to find this solution, which is

$$q(t) = C\epsilon + c_1 e^{\frac{-t}{RC}}.$$

Differential equations of this type and their solutions are studied in our freshman Calculus I course, MA104. This specific equation is studied in the second core physics course, PH202. Our hope in presenting it in the MST ILAP was to show a strong connection between physics and mathematics. The physics instructors are happy since their students gain a better understanding of the mathematics behind their applications, and the math instructors are happy because their students see real applications of the mathematics they teach.

This phase of the ILAP concluded with a preliminary data analysis of the RC Time data collected during the physics lab as outlined in section 2.1. Recall that each observation in the RC Time data set was the amount of time it took to achieve a certain voltage with a specified, constant resistance. Each group took 50 observations at this constant resistance. The cadets performed a traditional descriptive statistics analysis of this data which included determining certain sample statistics such as the mean, the median, and the variance, as well as creating and analyzing histograms, boxplots, and stem-and-leaf diagrams. The cadets were also required to identify and explain sources of variability and recommend procedures to reduce or control them. Additionally, they were asked to make a conjecture about the distribution underlying the RC Time data and then to use their sample statistics to estimate the parameters for the distribution they chose.

The tools the cadets were given to take these measurements were not ideal for the task and this caused a great deal of error. The device that measured voltage had a digital readout that jumped discretely a few times per second as the voltage in the circuit increased. It therefore usually did not hit the desired "stopping" voltage (the voltage where the time reading was to be taken) exactly, but rather jumped over it. In addition, manual stopwatches were used to measure the time from closing the circuit until the stopping voltage was achieved or exceeded, and this introduced even more error into the process. Since they had taken these measurements themselves, the cadets did an excellent job identifying these problems when asked to explain the variability in the data. Our experience is that similar students would find this type of question much harder if they were not given the opportunity to actually perform the experiments.

After completing the work described in sections 2.1 and 2.2, the cadets submitted a preliminary report containing their results and analysis. This allowed us to make sure each group was headed generally in the right direction and provide feedback as required. It also allowed us to collect the data the students would use for the statistical analysis described in section 2.4.

2.3 RLC Circuit

In this section of the ILAP, the cadets investigated a typical RLC circuit with a forcing function as shown in Figure 2. They had not been exposed to this type of circuit before, although they did have at least some limited experience with the associated differential equation.

The application of Kirchhoff's Loop Rule to the circuit in Figure 2 leads to the linear, second order, nonhomogeneous differential equation shown here in terms of current:

$$i'' + \frac{R}{L}i' + \frac{1}{LC}i = \frac{-V_m\omega}{L}\sin(\omega t).$$

where i is current in amperes, R is resistance in ohms, C is capacitance in farads, L is the inductance in henries, V_m is the maximum applied voltage in volts, and ω is the angular frequency in radians per second.

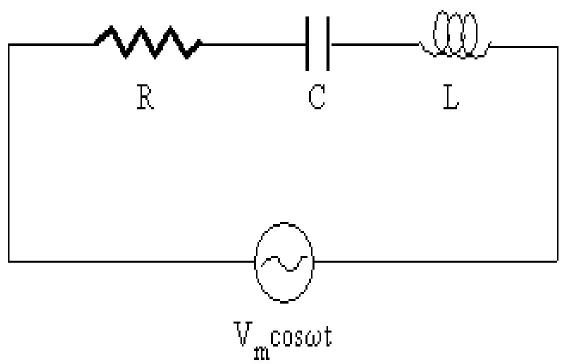

Figure 2: RLC Circuit

Depending on the values of R, L, C, and V_m, the solution to the homogeneous part of this differential equation can be either *underdamped*, *critically damped*, or *overdamped*. The primary focus of analysis from this point on was the underdamped case, where the current exhibits oscillating decay.

Next, the cadets determined the long term behavior for the underdamped case. Since the magnitude of the homogeneous portion of the solution decays exponentially, after a long time we are left with only the nonhomogeneous part, which is

$$i(t) = \frac{V_m L\omega(\omega^2 - \frac{1}{LC})}{L^2(\omega^2 - \frac{1}{LC})^2 + R^2\omega^2}\sin(\omega t) + \frac{V_m\omega^2 R}{L^2(\omega^2 - \frac{1}{LC})^2 + R^2\omega^2}\cos(\omega t).$$

Determining $i(t)$ is a cumbersome calculation to perform by hand. This would have been beyond the reach of most of the cadets if technology had not been available to them. We strongly encourage the use of technology to help in problem-solving, and the students were quick to enlist the help of a computer to attack this problem. However, many groups failed to recognize that the homogeneous portion decays; they had trouble with this problem even with the help of a computer algebra system.

In addition to determining the long term behavior of the RLC circuit analytically, the cadets investigated some other aspects of the circuit's behavior graphically. They were given numerical values $R = 200$ ohms, $L = 1$ henry, $C = 20$ microfarads, and $V_m = 5$ volts, and asked for some relevant plots and explanations. For example, the cadets were told to plot current vs. angular frequency and then use it to analyze the relationship between them. The plot is shown in Figure 3 below.

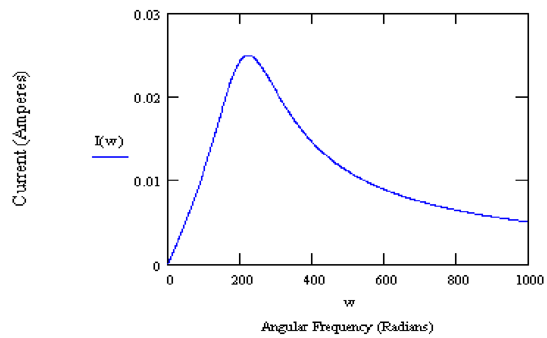

Figure 3: Current v. Angular Frequency

From the plot it appears that the maximum current occurs at some point just over $\omega = 220$ radians per second. This result nicely confirms the analytical approach that tells us that the critical frequency of $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(1)(0.00002)}} = 100\sqrt{5} \approx 223.6$ radians per second is where current is maximized.

2.4 Statistical Analysis

The remainder of the ILAP involved conducting statistical analyses on two data sets related to the initial experiment. First, they were given a set of "Radio Lifetime" data that represented the lifetimes of randomly selected military radios. The cadets had learned that these radios contain RLC circuits in their physics classes. This data set was created to approximately follow an exponential distribution. The cadets were asked to use Minitab to investigate this data with histograms, stem-and-leaf diagrams, and boxplots. They used Minitab's descriptive statistics macro to determine the sample mean, median, and variance of their data. At this point, they were able to conjecture that the Radio Lifetime data's distribution was approximately exponential. They had previously studied this distribution and knew its probability density function to be

$$f(t) = \begin{cases} \lambda e^{-\lambda t} & \text{for } t \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

They then used point estimation techniques to find an estimate for the distribution's parameter, λ. Once they obtained this value, they could answer probability questions about radios. These questions in turn were used to demonstrate the *memoryless* property of the exponential distribution, which states that

$$P(T \geq t + t_0 | T \geq t_0) = P(T \geq t).$$

The other data set, called "RC Mean", contained values that represented the average of the 50 observations for RC Time discussed in section 2.1. Each group reported the mean of their 50 observations as part of their preliminary report. We had planned to use each of the 200 or so student reported mean values as a data point for RC Mean. Thus if $X_1, X_2, ..., X_{50}$ are random variables representing the 50 RC Time observations, then each group reported their particular realization of $\bar{X}$, the sample average of those observations. We know that $E(\bar{X}) = \mu_{\bar{X}} = \mu$ and, assuming independence of the X_i's, $V(\bar{X}) = \sigma^2_{\bar{X}} = \sigma^2/n$. We had hoped it would be reasonable to treat every observation from every group as in some sense coming from the same population. This would allow us to invoke the central limit theorem and claim that RC Mean should behave like an approximately normally distributed random variable with mean $\mu_{\bar{X}}$ and variance $\sigma^2_{\bar{X}}$. As we had hoped, the resultant data set was generally bell-shaped and symmetrical. It also had sample mean and sample variance values that roughly agreed with what could be predicted from an individual group's RC Time data by using sample data to estimate μ and σ^2. Unfortunately, more sophisticated normality testing revealed to us that the data did not follow an approximately normal distribution. Only an examination of the higher moments (skewness and kurtosis) revealed this. This departure from normality was probably due to different groups conducting the initial RC Time observations outlined in section 2.1 with subtley different equipment and in subtley different ways.

One of the major objectives of this project was to have the cadets involved in the scientific process all the way from data collection to statistical analysis. Another objective was to *demonstrate* the central limit theorem in an actual experiment. Before the cadets submitted a preliminary report containing their RC Time data, we had no way of knowing if the subtle differences in each group's experimental conditions would be enough to prevent the RC Mean data from approximately following a normal distribution. Once they did report these values, and the results indicated that RC Mean was *not* approximately normal, we decided the best way to still attain both of the above objectives was to replace the RC Mean data set with random data from a normal distribution with the same mean and variance as the original RC Mean data. In other words, the RC Mean data was changed to appear as it might have if all $200 \times 50 = 10000$ RC Time data points had been collected by the same cadet with the same equipment. We discussed this change and the reasons for it with the cadets.

With RC Mean changed as indicated, the cadets could proceed as they had done with the Radio Lifetime data. They used Minitab to collect descriptive statistics and create plots. They used this information to determine estimates for the parameters $\mu_{\bar{X}}$ and $\sigma^2_{\bar{X}}$, which represent the mean and variance of RC Mean, respectively. They also were asked for confidence intervals for $\mu_{\bar{X}}$ and $\sigma^2_{\bar{X}}$. For the most part they relied on Minitab to do this for them, but they were also expected to verify that the necessary assumptions for calculating these intervals held rather than simply regurgitating the computer output.

3 SOME RESULTS AND FUTURE DIRECTIONS

Cadet performance on the MST ILAP was generally very good. They proved that they understood material from the mathematics and science courses they had studied over the past two years and could apply that knowledge to a large, multi-concept project. Their command of technological tools was especially noteworthy. Not only was software like Minitab and Mathcad used extensively, but the actual reports the cadets turned in were indicative of excellent presentation and communication skills. In other words, the cadets demonstrated that they could "understand and apply the mathematical, physical, and computer sciences to reason scientifically, solve quantitative problems, and use technology," as required by the mathematics, science, and technology goal.

The MST ILAP also revealed some weaknesses that should be addressed. The cadets were generally not able to solve the differential equation associated with the RLC circuit, at least not without significant help from instructors or other cadets. As identified in section 2.3, this was at least in part due to the fact that they did not recognize the need to ignore the homogeneous portion and concentrate on the nonhomogeneous portion of the solution. The students that did correctly solve this equation did so because they could *combine* a mastery of the technological tools (such as Mathcad) with a solid understanding of some of the mathematical concepts they had been taught. Successfully recalling and using these mathematical concepts was easier for students who maintained portfolios of the work they had done in previous courses.

We should point out that the RLC circuit portion of the ILAP was designed to stretch the cadets beyond the base of knowledge they had been taught, so it was not surprising that they had difficulty. From a pedagogical standpoint, as long as the questions are not too far beyond what the students have seen, there is a great deal of learning that occurs when they are asked to investigate a new topic. This learning takes place not only in the subject area under investigation, but more importantly in the area of learning about research or "learning how to learn."

In addition, cadets seemed to resist and fear some of the open-ended questions. They were generally not comfortable with questions which do not appear to them to have clear-cut answers. Although this is probably typical of college sophomores, it is particularly important at USMA to help the cadets work to overcome this resistance and anxiety. As Army leaders of the 21st century, they will need the ability to tackle ill-defined problems posed in complicated situations.

We plan to continue giving this type of project to the cadets at USMA. Our plan is to focus on a new partner department each year, refining the concept of the MST ILAP as we go. Instead of the Department of Physics, this year we plan to do a project with the Department of Civil and Mechanical Engineering. Although this project has not yet been fully developed, we generally intend to provide students with drawings of a concrete dam, along with some messy data the describes seasonal water levels. We will put the cadets in the role of young engineers who must decide if the dam meets certain criteria given the water data. Eventually, we hope to include more than two departments in each project. Our goal in the long term is to conduct a version of the MST ILAP every semester that involves mathematics, science, and technology from the entire spectrum of the disciplines taught at West Point. The project discussed in this paper as well as several others we have recently administered can be downloaded from http://www.dean.usma.edu/math/CORE/ma206/ by following the link to "Old Projects."

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BIOGRAPHICAL SKETCHES

Steve Horton is an Associate Professor in the Department of Mathematical Sciences at the United States Military Academy and a Lieutenant Colonel in the US Army. He received his BS degree from USMA in 1982, and MSOR and PhD degrees from the Georgia Institute of Technology in 1991 and 1997, respectively. His research interests include combinatorial optimization, graph theory, and computational complexity. In his spare time he plays basketball, guitar, and bridge.

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