MA386 - Introduction to Numerical Analysis
Homework Assignment 5, 100 points
Due In Class October 24

Please show all work. If you use a computer to help solve any problems, please included a printout of your work.

1. Section 4.1: 6 c. Specify the formulas you are using to calculate each derivative. Be careful at the endpoints (see equation (4.4) and the equation before it on page 171 if you're stumped).

2. Consider the formula below. It gives an approximation formula for $f'(x)$ at a point x_0 using 4 neighboring points. We are told that the method is $\mathcal{O}(h^m)$, but what is m? Determine the rate of convergence experimentally, using $f(x) = x^2 e^x$ at $x_0 = 1$. `Excel` would probably be a wonderful tool for this exercise... don't forget to include a plot of $\log(h)$ vs. $\log(error)$.

$$f'(x_0) = \frac{-2f(x_0 - h) - 3f(x_0) + 6f(x_0 + h) - f(x_0 + 2h)}{6h} + \mathcal{O}(h^m)$$

   NOTE: Don't let h get too small: in this case, less than 10^{-5}. The results beyond that won't be reliable!

3. In the above problem, there is a warning that we can't trust the approximations as h gets small. Give numerical evidence showing that the error does not decrease as h goes to zero. Briefly explain why this is the case.

4. Section 4.2: 5. With this problem you'll take a few "ok" approximations and use extrapolation to find a better one. Note that in this case there are only even powers of h in the error, so you'll have to derive a *new* extrapolation formula. Remember the essential steps:

   - Write down the formula $M = N_1(h) + K_1 h^2 + \ldots$ evaluated at $h/2$ instead of h.
   - Multiply it by the appropriate constant and subtract $M = N_1(h) + K_1 h^2 + \ldots$ to get rid of the lowest power of h.
   - When the dust settles, you can solve for M (the thing you're approximating). This will give you the new approximation scheme, $N_2(h)$.
   - You'll have to repeat this process a few times to get $N_4(h)$.

   Be sure to show all work.

5. Section 4.3: 18. Show all work. Does the resulting formula coincide with any methods we've studied? If so, which one?

6. BONUS - Using Taylor expansions and algebra trickery, prove that the approximation formula in problem 2 converges at (roughly) the rate you found experimentally.

7. BONUS[1] - With regard to problem 3: Following the analysis of example 3 on page 174, see if you can predict the h that will give minimum error. Hint: in this case, the error is bounded by the function

$$e(h) = \frac{\epsilon}{h} + \frac{h^3}{12}M,$$

   where ϵ is machine epsilon and M is a constant depending on our function f.

[1] This one is a little challenging; I think you need to do the one above before attempting it.