RESEARCH STATEMENT

EDWARD FUSELIER

1. Introduction

My area of research is approximation theory. Approximation theory can be described most briefly as the study of how well functions can be approximated by classes of other (usually simpler) functions. This definition is of course too simplistic; approximation theory is a very rich, active research area of both pure and applied mathematics.

With regard to applications, the current needs of the scientific community often lead to interesting, and sometimes purely mathematical, questions for approximation theorists. Presently there is demand for understanding approximations when one is given unstructured data coming from a very high-dimensional space. Handling such problems is difficult. For example, conventional approximation techniques, such as finite element methods, require one to create a mesh based on the data sites, which can be expensive to compute even in small dimensions (especially if one is dealing with domains having complicated geometry or ones with moving boundaries). These issues lead naturally to a need for so-called "meshless" methods. My current research area, radial basis functions (RBFs), is a promising meshless tool for handling multi-dimensional scattered data. For me, this field has just the right blend of mathematical beauty and practical application.

2. A Brief Overview of Radial Basis Functions

RBFs are probably best known for their applications to scattered data interpolation problems. Given a finite set of points $X \subset \mathbb{R}^n$ and data associated with each point, the objective is to find a continuous function that fits the data at the points. A *radial basis function* is a positive definite function $\phi : \mathbb{R}^n \to \mathbb{R}$ that happens to be radial, i.e. $\phi(x) = \phi(\|x\|)$. Given such a function, one searches for an interpolant that is a linear combination of shifts of ϕ. Concretely, we look for an interpolant from the following function space:

$$F_{\phi,X} := \operatorname{span}\{\phi(\cdot - x_j) \,|\, x_j \in X\},$$

i.e., ϕ generates a "basis" of the approximation space. If ϕ is an RBF, then a unique interpolant always exists, and the existence of an interpolant does not depend on the number or location of the data sites. Such functions do exist, and popular examples of RBFs include Gaussians, Hardy multiquadrics, thin plate splines, and Wendland functions. A good reference for RBFs is Wendland's book [13].

While RBFs were initially studied for interpolation problems on $\mathbb{R}^n$, the theory reveals several bonuses. First, RBFs can be generalized to provide approximation when data is collected on general manifolds, or even on locally-compact groups [3, 14]. Secondly, RBFs can also be "customized" to develop new approximation techniques. They can be modified to construct approximations when the data is obtained in

a variety of ways, whether it be by point evaluations (interpolation), derivative or integral information, or other useful linear functionals. Hence RBFs can be used to numerically solve partial differential equations, for example. Further, RBFs can be used to develop approximations that preserve certain physical properties of the underlying target function, e.g. being divergence-free or curl-free. These ideas can also be extended to divergence-free and curl-free approximation on manifolds. Much of my research thus far has addressed aspects of the mathematical development of the divergence-free and curl-free RBF theory.

3. My Results

This section contains various results stemming from my research.

3.1. **Motivation.** Many physical applications involve vector fields that are divergence-free ($\nabla \cdot f = 0$) or curl-free ($\nabla \times f = 0$), such as compressible fluids and magnetic fields, to name a few. To successfully construct approximants that have these properties, matrix-valued RBFs were introduced by Narcowich and Ward in 1994 [11]. They constructed matrix-valued functions that yield divergence-free interpolants at scattered points. Constructing such functions turns out to be fairly simple. If ϕ is a scalar-valued function consider

$$\Phi_{div} \quad := \quad \left(-\Delta I + \nabla\nabla^T\right)\phi,$$

where ∇ is the $n \times 1$ gradient operator and $\Delta = \nabla^T\nabla$ is the Laplacian operator. Then Φ_{div} is an $n \times n$ matrix-valued function with divergence-free columns. Further, if ϕ is an RBF, this function can be used to produce divergence-free approximations. A similar construction works for the curl-free case. Indeed, the kernel

$$\Phi_{curl} \quad := \quad -\nabla\nabla^T\phi$$

produces curl-free approximants.

It is natural at this point to ask how "close" an RBF approximation is to the unknown function generating the data, also known as the *target function*. This question has been historically answered following the "native space" framework. Each scalar RBF ϕ gives rise to a space of functions called the *native space* of ϕ, denoted by $\mathcal{N}_\phi$. These are Hilbert spaces for which ϕ is the reproducing kernel, i.e. $\mathcal{N}_\phi$ is endowed with an inner product $(\cdot,\cdot)_{\mathcal{N}_\phi}$ such that if $f \in \mathcal{N}_\phi$, then $(f, \phi(\cdot - y))_{\mathcal{N}_\phi} = f(y)$. There is an analogue to this for matrix-valued kernels. Native spaces tend to be small, so such error estimates can be quite limiting. Finding approximation estimates for functions outside the native space is sometimes referred to as "escaping" the native space. While I have worked on other aspects of the theory, the main focus of my work has been to escape the native space when using divergence-free or curl-free approximants.

The rest of the section contains a snapshot of my research. The first results involve characterizing the native space in terms of more concrete, well-known function spaces. Other results are centered around the numerical stability of the interpolation process which, somewhat surprisingly, is directly related to the ability to escape the native space. The error estimates follow. There are also a few theorems extending these ideas to tangent vector approximation on $\mathbb{S}^2$, where we also us a "doubling trick" to double the approximation rates when the target is smooth and give an interesting

inverse result. Finally, we give an RBF Helmholtz-Hodge decomposition technique on $\mathbb{S}^2$, which has potential geophysical applications.

3.2. **A Characterization of $\mathcal{N}_{\Phi_{div}}$.** It is useful to work with concrete, well-known function space norms instead of abstract native space norms. The following result, from [5], gives us another form of the native-space inner-product and ultimately allows us to realize native spaces as Sobolev spaces with equivalent norms. The Fourier and functional notation below is standard, we use A^+ to denote the pseudo-inverse of a matrix A, and vector functions will be given in bold-face.

Theorem 1. *Suppose that $\phi \in C^2(\mathbb{R}^n) \cap L_1(\mathbb{R}^n)$ is a positive definite function such that $\Delta\phi \in L_1(\mathbb{R}^n)$. Define*

$$\mathcal{G}_{div} := \left\{ \mathbf{f} \in L_2 \cap C : \int_{\mathbb{R}^n} \widehat{\mathbf{f}}^*(\xi)\widehat{\Phi}^+_{div}(\xi)\widehat{\mathbf{f}}(\xi)d\xi < \infty \text{ and } \xi^T\widehat{\mathbf{f}}(\xi) = 0 \text{ a.e.} \right\}$$

and equip this space with the bilinear form

$$(\mathbf{f}, \mathbf{g})_{\mathcal{G}_{div}} := (2\pi)^{-n/2} \int_{\mathbb{R}^n} \widehat{\mathbf{g}}(\xi)^* \widehat{\Phi}_{div}(\xi)^+ \widehat{\mathbf{f}}(\xi)d\xi.$$

Then $\mathcal{G}_{div}$ is a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{G}_{div}}$ and reproducing kernel Φ_{div}. Hence $\mathcal{G}_{div} = \mathcal{N}_{\Phi_{div}}$ and the inner products are the same.

There is of course an analogous curl-free theorem, which we omit here. As a corollary, if the Fourier transform of ϕ has algebraic decay the native spaces of Φ_{div} and Φ_{curl} are subspaces of Sobolev spaces. Indeed, if $\widehat{\phi}$ satisfies

$$C_1 \left(1 + \|\xi\|_2^2\right)^{-(\tau+1)} \leq \widehat{\phi}(\xi) \leq C_2 \left(1 + \|\xi\|_2^2\right)^{-(\tau+1)}, \tag{1}$$

then $\mathcal{N}_{\Phi_{div}}(\mathbb{R}^n)$ is the subspace of $H^\tau(\mathbb{R}^n)$ consisting of divergence-free functions with a vanishing moment. We denote the space of Sobolev functions with a vanishing moment by $\widetilde{H}^\tau(\mathbb{R}^n)$. The norm for this space, which given (1) is equivalent to the native space norm, is given by

$$\|\mathbf{f}\|^2_{\widetilde{H}^\tau(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \frac{\|\widehat{\mathbf{f}}(\xi)\|^2}{\|\xi\|_2^2} \left(1 + \|\xi\|_2^2\right)^{\tau+1} d\xi.$$

While it is not hard to show that this class of functions is Sobolev, it can be shown that the norm is not equivalent to a Sobolev norm on $\mathbb{R}^n$. However, if we restrict our attention to compact sets, then these spaces are Sobolev spaces with a norm *equivalent* to a Sobolev norm. This result is from [5].

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Then we have*

$$H^\tau(\Omega) = \left\{ f|_\Omega,\ f \in \widetilde{H}^\tau(\mathbb{R}^n) \right\}.$$

Furthermore, the norm $\|\cdot\|_{H^\tau(\Omega)}$ is equivalent to $\|\cdot\|_{\widetilde{H}^\tau(\mathbb{R}^n)}$.

Realistically we always approximate on compact domains, so these results imply that when ϕ satisfies (1) we can deal with standard Sobolev spaces.

3.3. **Stability.** Let $X \subset \mathbb{R}^n$ be a finite set of distinct points. The following result gives insight into the conditioning of the interpolation matrix for divergence-free kernels on X, denoted by $A_{X,\Phi_{div}}$, by estimating the norm of its inverse. Since the interpolation matrix is symmetric and positive definite, this amounts to bounding its lowest eigenvalue, $\lambda_{min}(A_{X,\Phi_{div}})$, from below. Estimates are typically found in terms of

$$q_X := \frac{1}{2} \min_{j \neq k} \|x_j - x_k\|,$$

which is known as the *separation radius* of X. The following result is from [5].

Theorem 3. *Let ϕ be an even positive definite function, which possesses a positive Fourier transform $\widehat{\phi} \in C(\mathbb{R}^n/0)$. With the function*

$$M(\sigma) := \inf_{\|\xi\|_2 \leq \sigma} \widehat{\phi}(\xi)$$

a lower bound on the smallest eigenvalue of the interpolation matrix is given by

$$\lambda_{\min}(A_{X,\Phi_{div}}) \geq \left(\frac{\sigma^2}{16\pi}\right)^{\frac{n}{2}+1} \frac{M(\sigma)\pi}{(4\pi)^n \Gamma\left(\frac{n+2}{2}\right)}$$

for any $\sigma > 0$ satisfying

$$\sigma \geq \widetilde{C}/q_X.$$

Modifying the proof of this slightly gives us the corresponding curl-free result, which we omit. Also, if ϕ has finite smoothness it can be shown that, in terms of the order of the separation radius, these stability estimates are the *best possible.*

3.4. **Error Estimates: Escaping the Native Space.** Now we give error estimates for functions that are too rough to be in the native space. The error estimates we give are in terms of the *mesh norm.* Given a compact set $\Omega \subset \mathbb{R}^n$ and a finite set $X \subset \Omega$, the mesh norm is given by

$$h_{X,\Omega} := \sup_{x \in \Omega} \inf_{x_j \in X} \|x - x_j\|_2.$$

Another important value in the estimate is the *mesh ratio*, given by $\rho_{X,\Omega} := h_{X,\Omega}/q_X$. In what follows, we will let $I_X\mathbf{f}$ be the divergence-free RBF interpolant to $\mathbf{f}$ on X if $\mathbf{f}$ is divergence-free, and the curl-free RBF interpolant to $\mathbf{f}$ on X if $\mathbf{f}$ is curl-free. Also, we assume that the Fourier transform of ϕ has algebraic decay as in (1). The following result is from [6] in dimension $n = 2$ or 3. For technical reasons, we assume that $\tau = k + s$ with $k > n/2$ and that $\beta > n/2$ is an integer.

Theorem 4. *Let τ and β be given as above with $\tau \geq \beta$, and let $\Omega \subset \mathbb{R}^n$ be compact, simply connected, and have $\mathcal{C}^{k,1}$ boundary. Then if $\mathbf{f} \in H^\beta(\Omega)$ is a divergence-free (curl-free) function, we have the estimate*

$$\|\mathbf{f} - I_X\mathbf{f}\|_{W_q^\mu(\Omega)} \leq C h_{X,\Omega}^{\beta-\mu-n(1/2-1/q)_+} \rho_{X,\Omega}^{\tau-\beta} \|\mathbf{f}\|_{H^\beta(\Omega)} \ \forall\ h_{X,\Omega} \leq C_\Omega,$$

where μ is any integer such that $0 \leq \mu < \beta - n/2$ and C is a constant independent of X and $\mathbf{f}$.

Note that if, as $h_{X,\Omega}$ gets smaller, the data sites are chosen so that $\rho_{X,\Omega} \leq C$ for some constant C, we get standard Sobolev estimates. We should also note that these results depend heavily on the existence of a divergence-free band-limited interpolant, which is interesting in its own right. The following result, and its curl-free counterpart, is from [6]. Here $\widetilde{H}^{\tau}_{div}$ is the subspace of $\widetilde{H}^{\tau}$ of divergence-free functions, and $\widetilde{\mathcal{B}}^{\sigma}_{div} \subset \widetilde{H}^{\tau}_{div}$ contains only functions whose Fourier transforms are supported in the ball of radius σ.

Theorem 5. *Let $\tau, t \in \mathbb{R}$ such that $\tau > n/2$ and $t > 0$. Given $\mathbf{f} \in \widetilde{H}^{\tau+t}_{div}$ and a point set $X \subset \mathbb{R}^n$ with separation distance q_X, there exists a function $\mathbf{f}_\sigma \in \widetilde{\mathcal{B}}^{\sigma}_{div}$ such that*

$$\mathbf{f}|_X = \mathbf{f}_\sigma|_X \text{ and } \|\mathbf{f} - \mathbf{f}_\sigma\|_{\widetilde{H}^\tau} \leq 5 \cdot \mathrm{dist}_{\widetilde{H}^\tau}(\mathbf{f}, \widetilde{\mathcal{B}}^{\sigma}_{div}) \leq 5\kappa^{-t} q_X^t \|\mathbf{f}\|_{\widetilde{H}^{\tau+t}}.$$

with $\sigma = \kappa/q_X$, where $\kappa \geq 1$ depends on only τ and n.

3.5. **Divergence-Free Approximation on Manifolds.** All results in this section are from [7], which is specific to divergence-free tangent fields on $\mathbb{S}^2$. However, the ideas can be extended to more general manifolds. In [12], Narcowich, Ward, and Wright introduced the kernel Ψ_{div}, which generates vector interpolants that are tangent to the sphere and have zero surface-divergence. Given an RBF ϕ, we can construct Ψ_{div} via

$$\Psi_{div}(x, y) := Q_x^T \left(-\nabla\nabla^T \phi(x, y)\right) Q_y,$$

where Q_p is the 3×3 matrix such that $Q_p z = p \times z$. Given a tangent vector field $\mathbf{f}$ sampled on a data set $X \subset \mathbb{S}^2$, one can construct an interpolant of the form

$$I_X \mathbf{f}(x) = \sum_{j=1}^{N} \Psi_{div}(x, x_j)\, \mathbf{s}_j,$$

where $\mathbf{s}_j$ is tangent to the sphere at x_j. Sharp stability estimates for these kernels were given in [7], but we omit them here. It is interesting to note that both the stability estimates, and error estimates given below, depended heavily on the curl-free results on $\mathbb{R}^3$.

For the rest of this section, we assume that ϕ is an RBF satisfying

$$C_1 \left(1 + \|\xi\|_2^2\right)^{-(\tau+3/2)} \leq \widehat{\phi}(\xi) \leq C_2 \left(1 + \|\xi\|_2^2\right)^{-(\tau+3/2)}, \tag{2}$$

Below is an error estimate for Sobolev functions too rough to be in the native space of Ψ_{div}. Here, for technical reasons, we assume that $\tau = k + s$ with $0 \leq s < 1$ and $k > n/2$ is an integer.

Theorem 6. *Let $\tau \geq \beta > 1$ and let ϕ be an RBF satisfying (2). Also, let $X = \{x_j\}_{j=1}^N \subset \mathbb{S}^2$ be a set of distinct points with mesh norm h_X, separation radius q_X and mesh ratio $\rho_X = h_X/q_X$. If $\mathbf{f} \in H^{\beta}_{div}(\mathbb{S}^2)$, and if μ is an integer such that $0 \leq \mu \leq \lfloor \beta \rfloor - 1$, we have*

$$\|\mathbf{f} - I_X \mathbf{f}\|_{W_q^\mu(\mathbb{S}^2)} \leq C \rho_X^{\tau-\beta} h_X^{\beta-\mu-2(1/2-1/q)_+} \|\mathbf{f}\|_{H^\beta(\mathbb{S}^2)}.$$

There are additional results stemming from the fact that we are working on the sphere. Using a "doubling trick," we can show that the approximation rates are doubled for very smooth functions.

Theorem 7. *Let τ be given as above, μ be an integer satisfying $0 \le \mu < k-1$, and let $\mathbf{f} \in H^{2\tau-\mu}_{div}(\mathbb{S}^2)$. Then*

$$\|\mathbf{f} - I_X\mathbf{f}\|_{H^\mu(\mathbb{S}^2)} \le C h_X^{2(\tau-\mu)} \|\mathbf{f}\|_{H^{2\tau-\mu}(\mathbb{S}^2)}.$$

Finally, we have an inverse result. That is, given a specific approximation rate, we can completely characterize the class of functions that our interpolants approximate at that rate. Here we must assume that all node sets involved are ρ-uniform, i.e. the mesh ratio is bounded uniformly for all node set.

Theorem 8. *Let $\tau > 1$ and let ϕ be an RBF satisfying (2). In addition, let $\mathcal{F}$ be a ρ-uniform family. If for some continuous vector field $\mathbf{f}$ there are constants $0 < \mu \le \tau$ and $c_{\mathbf{f}} > 0$ such that*

$$\|\mathbf{f} - I_X\mathbf{f}\|_{L_2(\mathbb{S}^2)} \le c_{\mathbf{f}} h_X^\mu$$

holds for all $X \in \mathcal{F}$, then, for every $0 \le \beta < \mu$, $\mathbf{f} \in H^\beta_{div}(\mathbb{S}^2)$.

3.6. **Vector Decomposition on $\mathbb{S}^2$.** According the the well-known Hodge decomposition, every function on the sphere can be decomposed into two orthogonal components: one is divergence-free and one is curl-free. Given a vector field tangent to $\mathbb{S}^2$, we will denote this decomposition as

$$\mathbf{f} = \mathbf{f}_{div} + \mathbf{f}_{curl}.$$

In [8], a scheme was created that will allow us to approximate each of these components simultaneously given only point evaluations of $\mathbf{f}$.

Naturally, there is a curl-free counterpart the the divergence-free SBF, which we denote by Ψ_{curl}. Given this and Ψ_{div}, we can construct a new kernel

$$\Psi := \Psi_{div} + \Psi_{curl}.$$

Using shifts of Ψ, be build an interpolant to $\mathbf{f}$ of the form

$$I_X\mathbf{f}(x) = \sum_{j=1}^N \Psi(x, x_j)\, \mathbf{s}_j,$$

where $\mathbf{s}_j$ is tangent to the sphere at x_j. Now the approximant is (quite easily) decomposed as follows

$$I_X\mathbf{f}(x) = \sum_{j=1}^N \Psi(x, x_j)\, \mathbf{s}_j = \underbrace{\sum_{j=1}^N \Psi_{div}(x, x_j)\, \mathbf{s}_j}_{\text{Div. free}} + \underbrace{\sum_{j=1}^N \Psi_{curl}(x, x_j)\, \mathbf{s}_j}_{\text{Curl free}}.$$

As it turns out each term of the decomposition approximates its counterpart in the Hodge decomposition of the target function. The following result is from [8].

Theorem 9. *Let $\tau \ge \beta > 1$ and let ϕ be an RBF satisfying (2). Also, let $X = \{x_j\}_{j=1}^N \subset \mathbb{S}^2$ be a set of distinct points with mesh norm h_X, separation radius q_X and mesh ratio $\rho_X = h_X/q_X$. If $\mathbf{f} \in H^\beta(\mathbb{S}^2)$, then for $0 \le \mu \le \lfloor \beta \rfloor - 1$ we have the following error estimates*

$$\begin{aligned} \|\mathbf{f}_{div} - (I_X\mathbf{f})_{div}\|_{H^\mu(\mathbb{S}^2)} &\le C\rho_X^{\tau-\beta} h_X^{\beta-\mu} \|\mathbf{f}\|_{H^\beta(\mathbb{S}^2)} \\ \|\mathbf{f}_{curl} - (I_X\mathbf{f})_{curl}\|_{H^\mu(\mathbb{S}^2)} &\le C\rho_X^{\tau-\beta} h_X^{\beta-\mu} \|\mathbf{f}\|_{H^\beta(\mathbb{S}^2)}. \end{aligned}$$

Furthermore, if $\mathbf{f} \in H^{2\tau-\mu}(\mathbb{S}^2)$ *we have the estimates*

$$\begin{aligned} \|\mathbf{f}_{div} - (I_X\mathbf{f})_{div}\|_{H^\mu(\mathbb{S}^2)} &\leq Ch_X^{2(\tau-\mu)}\|\mathbf{f}\|_{H^{2\tau-\mu}(\mathbb{S}^2)} \\ \|\mathbf{f}_{curl} - (I_X\mathbf{f})_{curl}\|_{H^\mu(\mathbb{S}^2)} &\leq Ch_X^{2(\tau-\mu)}\|\mathbf{f}\|_{H^{2\tau-\mu}(\mathbb{S}^2)}. \end{aligned}$$

4. Future Work

My current interests can be organized into three areas: vector approximation on manifolds, problems concerning scalar RBFs, and applications.

4.1. **Vector Approximation on Manifolds.** A few problems for the short-term are as follows. The use of smooth kernels for tangent vector types of approximations lead to several interesting results; the theory thus far only addresses kernels that have finite smoothness. Also, current projects include a poloidal/toroidal RBF decomposition that should be of interest in magneto-hydrodynamics [1]. As more of a long-term interest, we note that some problems require one to model flow across manifolds other than the sphere, and these manifolds might even change in time. Tangential RBFs could prove useful in modeling such problems.

4.2. **Scalar RBF Topics.** RBFs can be used to generate finite difference or quadrature methods, and the numerical results are encouraging. To be numerically feasible, the weights should satisfy certain conditions, which are currently under investigation. Also, it is well known that matrices associated with RBFs are ill-conditioned. We are currently investigating a construction for a preconditioners when one is working on the sphere. As more of a long-term goal, we note that error estimates for rough functions when Gaussian RBFs are used have not been given, and would be of much interest.

I'm also interested in RBFs applied to transport problems, where an RBF approximation can be used to model the mass (or mass density). Numerical solutions to transport schemes are notoriously unstable and difficult to approximate. However, recent work shows that RBFs might be useful for such approximations. Lorentz, Narcowich and Ward were able to derive error estimates for a simple example on the circle [9]. More recently, Flyer and Wright investigated transport on the sphere [4]. They were able to prove eigenvalue stability for linear advection on the sphere, and provided several promising numerical examples of the more difficult problem of deformational flow. These results are very encouraging, and the area is wide open for a rigorous theoretical framework.

4.3. **Applications.** As far as direct applications in my own research, the curl-free RBFs have been used successfully in an astrophysical application related to dark matter [2]. However, there is much potential for applications of divergence-free kernels both on $\mathbb{R}^n$ and the sphere, the most obvious of which would be applications to modeling fluids [10]. Also, applications of tangential vector RBFs to geophysics are currently under investigation.

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