Test for Uniformity Based on Convolutions of the Uniform Random Variable

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ABSTRACT

We present a test statistic for uniformity based on the convolution of iid uniform random variables. By using the software package A Probability Programming Language, we are able to find the exact distribution of the statistic for sample sizes thru $n = 50$, thus exact significance levels are attainable. The test statistic is designed with the purpose of finding shifts in the mean of an underlying distribution, relying on the probability integral transformation to create the uniform data. Monte Carlo simulation is used to show a significant increase in power over the benchmark Anderson–Darling test statistic. In this paper we introduce the statistic so that in a companion paper we can show the statistic's use to support further research in assisting in the planning and improvement of power of sequential censored life-testing.

Keywords: Computational Algebra Systems, Exact Distributions, Conditional Order Statistics, Lifetesting.

1 INTRODUCTION AND LITERATURE REVIEW

Testing for uniformity in a sample has many applications, many of which are explained in Chapter 8 of *Goodness-of-fit Techniques*, (D'Agostino and Stephens, 1988). One specific application comes from lifetesting, when the components have a reliability distribution that is continuous, but not necessarily uniform. Given a set of continuous random variates, $x_1, x_2, \ldots, x_n$ and a null hypothesis that the data comes from the underlying distribution X which has cdf $F(x)$, it is very well known that under the null hypothesis, $U = F(X)$ has the standard uniform distribution, $U(0,1)$. This result, known as the probability integral transform (PIT), can be used in transforming data to the uniform distribution so that uniformity tests can be performed for various purposes.

We present a test for uniformity that, relying on the PIT, will help us determine if the mean of experimental data, μ_a has shifted from that of μ_0. The Anderson–Darling statistic,

$$\text{A–D} = -n - \frac{1}{n}\sum_{i=1}^{n}[(2i-1)\ln(u_{(i)}) + (2n-1-2i)\ln(1-u_{(i)})],$$

is widely used because of its generally high power, compared to other uniformity test statistics, when looking for changes in the mean (Stephens, 1974). Our purpose is to find a test statistic that gives high power, preferably a higher power than the A–D statistic. With the availability of A Probability Programming Language (APPL) (Glen, et. al., 2001), we are able to find exact distributions of certain functions of random variables, specifically in this case the convolution of iid $U(0,1)$ rvs, and use these distributions to devise test statistics to find shifts in the mean of the underlying data. We present some characteristics of our statistic and discuss our eventual objective, that of improving sequentially censored lifetesting statistical inference, research reported in a companion paper.

2 THE TEST STATISTIC

The test statistic we propose, T_n, has the distribution of the convolution of iid $U(0,1)$ rvs:

$$T_n = \sum_{i=1}^{n} U_i = \sum_{i=1}^{n} F(X_i).$$

Prior to settling on this statistic, we explored other functions of iid $U(0,1)$ rvs. One option we explored was finding the distribution of $C = \sum_i^n \csc(U_i)$, as the cosecant function magnifies the statistic when the tails are too fat. The magnification happens at a quicker rate than that of the $-\ln(U)$, and we found this statistic had slightly higher power than A–D, when testing for shifts in σ_a away from σ_0. The statistic had appreciably less power, though, when testing for changes in μ, a fact geometrically understandable, as the changes in μ do not exaggerate the test statistic quickly. Furthermore, the exact distribution of C could not be found, and critical points had to be estimated with Monte Carlo simulation. We also considered $\min(U_i)$ and $\sum_{i=1}^n \tan(U_i)$ and found similar results.

We found considerable success with the test statistic $T_n = \sum_{i=1}^n U_i$ as a test for uniformity. Finding the distribution of the convolution of n iid $U(0,1)$ random variables becomes intractable by hand, once $n > 4$. However, by using APPL, we are able to determine the exact distribution of T_n for reasonable sample sizes. The distribution of T_n has n segments describing the PDF. Thus, the distribution of T_2 (the standard triangular distribution) has two segments, T_3 has three, and so on. As an example of a complete PDF, the distribution of T_7 is the following:

$$f(x) = \begin{cases} \frac{1}{720}\,x^6 & 0 < x < 1 \\ \frac{7}{120}\,x - \frac{7}{48}\,x^2 + \frac{7}{36}\,x^3 - \frac{7}{48}\,x^4 + \frac{7}{120}\,x^5 - \frac{1}{120}\,x^6 - \frac{7}{720} & 1 < x < 2 \\ \frac{1337}{720} - \frac{133}{24}\,x + \frac{329}{48}\,x^2 - \frac{161}{36}\,x^3 + \frac{77}{48}\,x^4 - \frac{7}{24}\,x^5 + \frac{1}{48}\,x^6 & 2 < x < 3 \\ -\frac{12089}{360} + \frac{196}{3}\,x - \frac{1253}{24}\,x^2 + \frac{196}{9}\,x^3 - \frac{119}{24}\,x^4 + \frac{7}{12}\,x^5 - \frac{1}{36}\,x^6 & 3 < x < 4 \\ \frac{59591}{360} - \frac{700}{3}\,x + \frac{3227}{24}\,x^2 - \frac{364}{9}\,x^3 + \frac{161}{24}\,x^4 - \frac{7}{12}\,x^5 + \frac{1}{48}\,x^6 & 4 < x < 5 \\ -\frac{208943}{720} + \frac{7525}{24}\,x - \frac{6671}{48}\,x^2 + \frac{1169}{36}\,x^3 - \frac{203}{48}\,x^4 + \frac{7}{24}\,x^5 - \frac{1}{120}\,x^6 & 5 < x < 6 \\ \frac{117649}{720} - \frac{16807}{120}\,x + \frac{2401}{48}\,x^2 - \frac{343}{36}\,x^3 + \frac{49}{48}\,x^4 - \frac{7}{120}\,x^5 + \frac{1}{720}\,x^6 & 6 < x < 7 \end{cases}$$

Interestingly, the PDF of T_{50} requires 91 pages of ASCII text to express. The distribution of T_n for $n \leq 50$ can be found at the first author's web site, `www.dean.usma.edu/math/people/glen`. Since the distribution of T_n is exactly known, the exact critical values are calculable, and exact significance levels are attainable for any sample of data. Table 1 shows the critical values used for our power simulations. For example the lower critical point for $\alpha = 0.05$ and sample size 10 is listed as 3.49611347, but can be found to at least 40 digits in Maple, Version 7 (Maple 2001). The test procedure is simply this: receive the failure data $(x_1, x_2, \ldots, x_n)$,

find $\sum_{i=1}^{n} F_X(x_i) = T_n$, then calculate p-value of the statistic using APPL, need for its ability to produce the exact distribution of convolutions of iid U(0,1) rvs up to size $n = 50$.

Table 1: Estimated critical values for T_n at various sample sizes and levels of significance

n	0.05	0.95
5	1.43464988	3.56535012
10	3.49611347	6.50388653
15	5.65912370	9.34087630
20	7.87489672	12.1250937
25	10.1244286	14.8755701
30	12.3979591	17.6020409
35	14.6896782	20.3103218
40	16.9958021	23.0041979
45	19.3137005	25.6862995
50	21.6414577	28.3579250

3 POWER COMPARISONS

As mentioned before, we are interested in finding shifts in μ_0 given the all parameters known null hypothesis that X has cdf $F(x)$. As our intended purpose for the test statistic is to assist in the lifetesting task of finding changes in mean lifetime of items on test, this power simulation varies μ_a from μ_0, but fixes the standard deviation, where possible. We fixed σ where possible to isolate the power of the test to find shifts in μ, as changes in mean lifetime of systems are often most relevant. We chose the Normal, Exponential and Gamma distributions as parent distributions for our null hypotheses. In the case of the Normal and Gamma distributions, we fixed $\sigma_0 = 1$ and only varied μ above and below μ_0. In the case of the Exponential, we have no choice but to vary μ_a and σ_a, as both depend on the parameter λ. We were interested in performing a similar simulation with the Weibull distribution, however, fixing σ and varying μ is not always possible, since the Weibull parameters are not readily solved for in such a manner, as not all combinations of mean and variance are possible

with the Weibull distribution. Table 2 gives the various parameters, means, and variances, for each simulation of the Normal, Exponential, and Gamma null hypothesis distribution.

Table 2: Distribution families, parameters, mean and variances for Monte Carlo Simulation

Normal Distribution, $H_0 : \mu = 1$, fixed $\sigma = 1$										
μ_a	-1	-0.8	-0.6	-0.4	-0.2	0.2	0.4	0.6	0.8	1
Exponential Distribution, $H_0 : \lambda = \mu = 1$										
$\lambda_a = \frac{1}{\mu_a}$	0.4	.6	0.7	0.8	0.9	1.25	1.5	1.9	2.3	2.7
Gamma Distribution, $H_0 : \alpha = \mu = 2.1$, $\beta = 4.41$ fixed $\sigma = 1$										
$\alpha_a = \mu_a$	1.1	1.3	1.5	1.7	1.9	2.3	2.5	2.7	2.9	3.1
β_a	1.21	1.69	2.25	2.89	3.61	5.29	6.25	7.29	8.41	9.61

Figures 1, 2, and 3 give the graphical results of our simulation. We chose sample sizes of $n = 5$ and $n = 25$ and compared the results of the A–D and T_n tests. The results show significant improvement in power for all three distributions. For example, in the Normal power experiment, where $n = 5$ and $\mu_a = 0.6$, the power for A–D is 26% and the power for T_n is 37%, a 42% increase in power. Similarly in the Gamma experiment, where $n = 25$ and $\mu_a = 2.3$, the power increase is from 17% to 28%, a 64% increase in power over the A–D statistic.

An important note involves the complexity of this Monte Carlo simulation. The simulation requires, among other things, the PIT on the Gamma distribution. Of course, the CDF of the Gamma is not in closed form. Currently the simulation is only possible by the APPL procedures working in the Maple (Maplesoft 2002) environment. For example, in order to accomplish the PIT, APPL is able to create the CDF of the null-hypothesis gamma distribution with $\alpha = \mu = 2.1$ and $\beta = 4.41$, which has the form

$$\begin{aligned} F(x) \quad = \quad & \tfrac{100000}{66987458901}\, x^{\frac{41}{200}} 21^{\frac{41}{200}} 10^{\frac{159}{200}} e^{-\frac{21}{20}x} [82181\, \texttt{WhittakerM}(\tfrac{41}{200}, \tfrac{141}{200}, \tfrac{21}{10}\, x) \\ & +143220\, \texttt{WhittakerM}(\tfrac{41}{200}, \tfrac{141}{200}, \tfrac{21}{10}\, x)x + 132300\, \texttt{WhittakerM}(\tfrac{41}{200}, \tfrac{141}{200}, \tfrac{21}{10}\, x)x^2 \\ & -71610\, x\, \texttt{WhittakerM}(-\tfrac{159}{200}, \tfrac{141}{200}, \tfrac{21}{10}\, x) - 82181\, \texttt{WhittakerM}(-\tfrac{159}{200}, \tfrac{141}{200}, \tfrac{21}{10}\, x) \\ & +92610\, x^3\, \texttt{WhittakerM}(\tfrac{41}{200}, \tfrac{141}{200}, \tfrac{21}{10}\, x) - 44100\, x^2\, \texttt{WhittakerM}(-\tfrac{159}{200}, \tfrac{141}{200}, \tfrac{21}{10}\, x)]/\Gamma(\tfrac{41}{100}), \end{aligned}$$

$0 < x$, relying on Maple's `WhittakerM` function, a solution to a differential equation. As a result, each dot on the charts in Figures 1, 2, and 3, represents $n = 5000$ samples, a

substantial amount of computing time, considering the simulation was done in Maple.

4 CONCLUSIONS AND FURTHER RESEARCH

In this paper we introduce T_n as an important test statistic for uniformity for the class of tests relying on the PIT to transform continuous lifetest data. The statistic has the advantages of higher power that of the benchmark A–D statistic and also an exactly known distribution, thus p-values for samples can be found precisely. Our intent is to explain, in a companion paper, the use of the T_n in sequential censored lifetesting. We will present a methodology that gives statistical inference to censored tests by combining conditional order statistic distributions with the T_n statistic to provide exact p-values real-time during the lifetest experiement. This method will provide test designers some options in designing lifetests that achieve necessary levels of statistical power, while minimizing the costs of running the tests.

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Figures

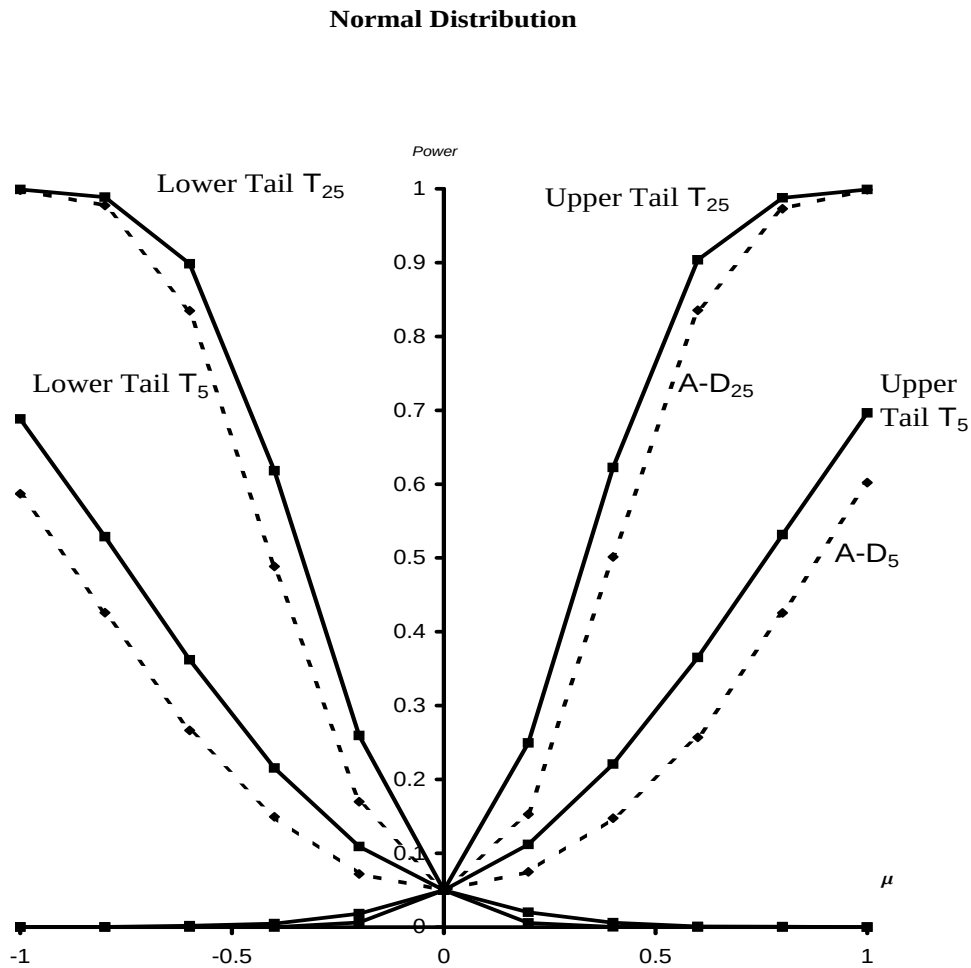

Figure 1: Results of Monte Carlo power simulation with underlying normally distributed data, $\sigma = 1$ and type I error $\alpha = 0.05$. Under H_0, $\mu = 0$. Note the T_n statistic enjoys significantly higher power than does the A–D statistic, expecially when μ_a is close to μ_0.

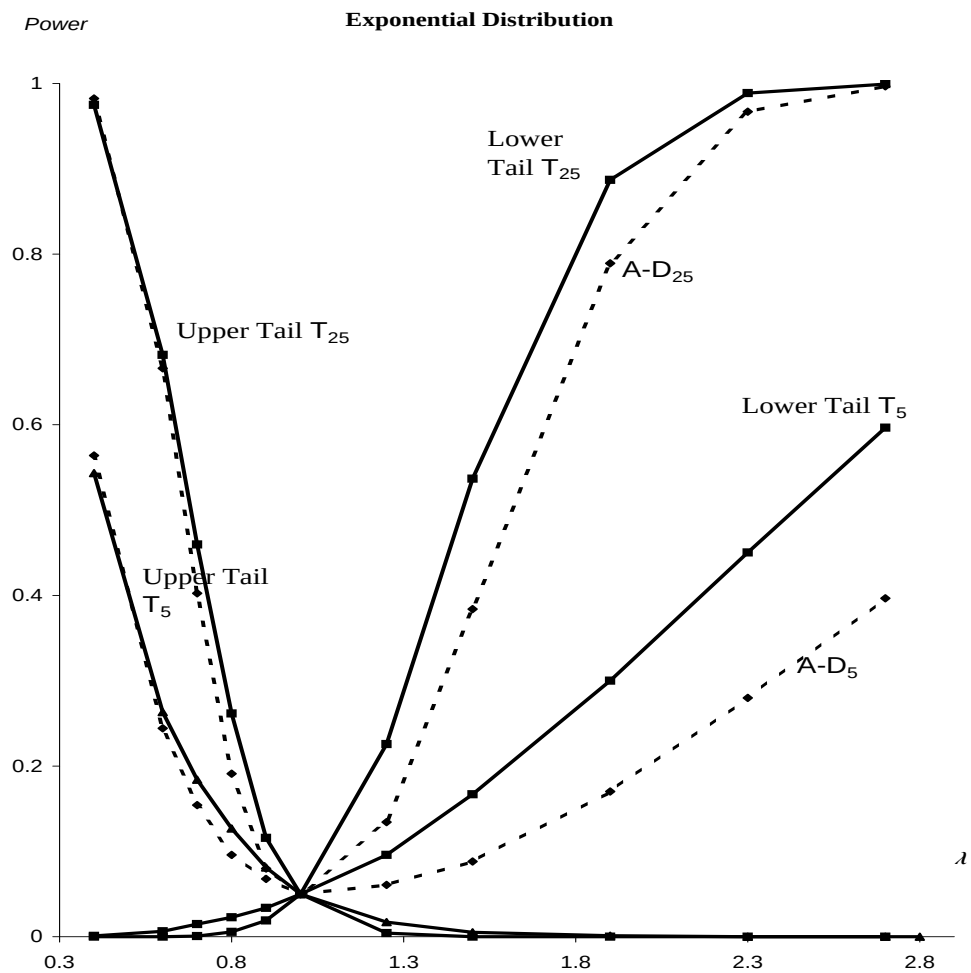

Figure 2: Results of Monte Carlo power simulation with underlying exponential distributed data with type I error $\alpha = 0.05$. Under H_0, the exponential distribution has parameter $\lambda = \frac{1}{\mu} = 1$. Thus, the upper tail test applies to the lower λ_a values. Again, the T_n enjoys substantially higher power than the A–D statistic.

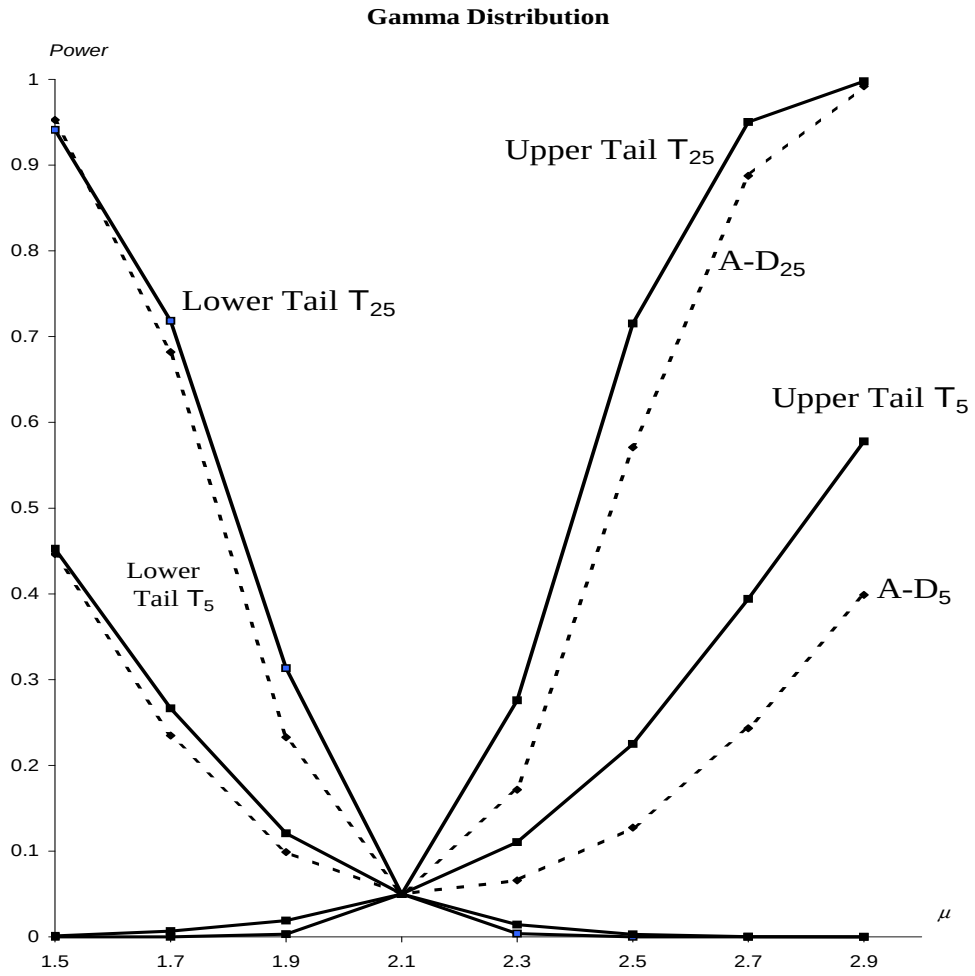

Figure 3: Results of Monte Carlo power simulation with underlying gamma distributed data, $\sigma = 1$ and type I error $\alpha = 0.05$. Under H_0, the gamma distribution has parameters $\alpha = \mu = 2.1$ and $\beta = 4.41$. Again, the T_n enjoys substantially higher power than the A–D statistic.