Computing the Distribution of the Product of Two Continuous Random Variables

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We present an algorithm for computing the probability density function of the product of two independent random variables, along with an an implementation of the algorithm in a computer algebra system. Some examples demonstrate the algorithm's application.

Keywords: algorithms, computational algebra systems, probability.

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1 Introduction

Rohatgi's well-known result (1976, page 141) for determining the distribution of the product of two random variables is straightforward to derive, but difficult to implement. Let X and Y be continuous random variables with joint PDF $f_{X,Y}(x, y)$. The PDF of $V = XY$ is

$$f_V(v) = \int_{-\infty}^{\infty} f_{X,Y}\left(x, \frac{v}{x}\right) \frac{1}{|x|}\, dx\,.$$

The implementation of this result, however, is not straightforward for general X and Y. Springer (1979) presents a chapter on finding distributions of products of random variables, relying mostly on Laplace and Mellin transformation techniques, as implementation of the earlier result is often too cumbersome. Difficulties occur as a result of both the myriad of variations to the limits of integration and the propensity of the PDF of V to be defined in a piece-wise manner. In this paper, we consider the cases when X and Y are independent and may have probability density functions (PDFs) defined in a piece-wise fashion. We present an algorithm that handles these difficulties and is implemented using the Maple computer algebra system.

2 Theorem

A simple theorem is presented in this section which illustrates some of the issues associated with a general algorithm for determining the PDF of the product of two independent random variables. For simplicity, assume that the random variable X has support on the interval (a, b) and the random variable Y has support on the interval (c, d). Also, the product space of the two random variables is assumed to fall entirely in the first quadrant.

Theorem. Let X be a random variable of the continuous type with PDF $f(x)$ which is defined and positive on the interval (a, b), where $0 < a < b < \infty$. Similarly, let Y be a random variable of the continuous type with PDF $g(y)$ which is defined and positive on the

interval (c, d), where $0 < c < d < \infty$. The PDF of $V = XY$ is

$$h(v) = \begin{cases} \int_a^{v/c} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & ac < v < ad \\ \int_{v/d}^{v/c} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & ad < v < bc \\ \int_{v/d}^{b} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & bc < v < bd \end{cases}$$

when $ad < bc$,

$$h(v) = \begin{cases} \int_a^{v/c} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & ac < v < ad \\ \int_{v/d}^{b} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & ad < v < bd \end{cases}$$

when $ad = bc$, and

$$h(v) = \begin{cases} \int_a^{v/c} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & ac < v < bc \\ \int_{a}^{b} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & bc < v < ad \\ \int_{v/d}^{b} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & ad < v < bd \end{cases}$$

when $ad > bc$.

Proof. Only the case of $ad < bc$ is considered. The other cases are proven analogously. Using the transformation technique (Hogg and Craig, 1995, page 173), the dummy transformation $Z = X$ and the transformation $V = XY$ constitute a 1–1 mapping from $\mathcal{A} = \{(x, y)|a < x < b, c < y < d\}$ to $\mathcal{B} = \{(z, v)|a < z < b, cz < v < dz\}$. Let u denote the transformation and w the inverse transformation. The transformation, inverse, and Jacobian are:

$$\begin{aligned} z &= u_1(x, y) = x & x &= w_1(z, v) = z \\ v &= u_2(x, y) = xy & y &= w_2(z, v) = v/z \end{aligned}$$

$$J = \begin{vmatrix} 1 & 0 \\ -v/z^2 & 1/z \end{vmatrix} = 1/z.$$

The joint PDF of Z and V is

$$f_{Z,V}(z, v) = f(w_1(z, v))g(w_2(z, v))\,|J| \qquad (z, v) \in \mathcal{B},$$

or

$$f_{Z,V}(z, v) = f(z)g(v/z)\frac{1}{z} \qquad (z, v) \in \mathcal{B}.$$

Integrating with respect to z over the appropriate intervals and replacing z with x in the final result yields

$$h(v) = \begin{cases} \int_a^{v/c} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & ac < v < ad \\ \int_{v/d}^{v/c} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & ad < v < bc \\ \int_{v/d}^{b} g\left(\frac{v}{x}\right) f(x) \frac{1}{x}\, dx & bc < v < bd \end{cases},$$

as desired. □

The geometry associated with the transformation is shown in Figures 1 and 2. Note that the transformation maps A_i onto B_i for $i = 1, 2, 3$. Although the transformation technique has been used to prove this theorem, the cdf technique could also have been used.

3 Implementation

The theorem in the previous section illustrates the importance of considering the magnitudes of the product of the coordinates of the southeast and northwest corners of the product space [i.e., (b, c) and (a, d)] when it lies entirely in the first quadrant. In order to apply the theorem to any continuous random variables X and Y, three generalizations need to be addressed.

1. Analogous theorems must be written for the cases when the (a, b) by (c, d) rectangle lies wholly in one of the other three quadrants.

2. Instead of having PDFs which are specified by a single standard function over their entire support, the random variables X and Y may be defined in a piece-wise manner over several intervals, forming many segments to the PDF (e.g., the triangular distribution).

3. The cases when 0 and $\pm\infty$ belong to the endpoints of the intervals which constitute the support of X and Y must be considered.

These generalizations result in 24 different cases that must be considered in order to correctly compute the limits of integration of the theorem. The paragraphs below address these cases. For quadrants II, III and IV, the limits of integration must be set appropriately based on the geometry of the transformation.

For random variables that are defined piece-wise over various intervals, let n be the number of intervals for X and let m be the number of intervals for Y. There are mn rectangular "product spaces" and the contribution of each to the value of the PDF of $V = XY$ must be computed. Furthermore, each "product space" can contribute differently to the PDF of V on up to three segments of the support of V. As a result, the PDF of V tends to become complicated very quickly, with an upper limit of $3mn$ segments to its PDF. Furthermore, each of these segments fits into one of the 24 cases of the integration mentioned above. For example, the product of two $U(0,1)$ random variables yields a random variable V with only one segment (see Example 4.1). But with only a slight change, e.g., $X \sim U(1,2)$ and $Y \sim U(3,4)$, yields a $V = XY$ defined differently on three segments (see Example 4.2).

The case where the support of a random variable contains 0 (e.g., $U(-1,2)$) poses special difficulty since some of the rectangular product spaces will not lie wholly in any one quadrant and cannot be handled by the previously developed techniques. Our solution to this difficulty is to add 0 as one of the endpoints of the intervals for X and Y whenever this case occurs, producing redundant segments, i.e., two segments on either side of zero with the same formula for the PDF.

The algorithm consists of a set-up portion, followed by nested loops that determine the contribution to the PDF of $V = XY$ separately for each of the four quadrants. The appendix contains the set-up portion and the algorithm associated with the first quadrant. The algorithm for the other quadrants is similar.

The set-up phase begins by setting n and m to the number of intervals that form the support of X and Y. Next, 0 is added as an interval delimiter for X and/or Y if the random variable can assume both positive and negative values, and 0 is not already an interval delimeter. Finally, the endpoints of the intervals which form the support of V are determined by taking all products of the endpoints of the X intervals times the endpoints of the Y intervals.

A nested set of loops follows that treats all pairings of X and Y intervals. As shown in Figure 1, the coordinates (a, c) are assigned to the southwest corner of the current rectangle of interest, and the coordinates (b, d) are assigned to the northeast corner of the current rectangle of interest. A test to determine which quadrant contains the current rectangle

is made at this point. Adding 0 as an interval delimiter in the set-up phase assures that the current rectangle will be completely contained in just one of the quadrants. Once the quadrant is determined, tests on c and d determine which integrals should be computed and the appropriate limits of integration. Finally, the insertion of 0 sometimes leads to a PDF for V with the same formula on both sides of 0. If this occurs, the program simplifies the PDF by removing 0 as an interval endpoint if the function is defined at 0.

4 Examples

This section contains applications of using the Maple procedure `Product`, which is our implementation of the algorithm described in the previous section.

Example 4.1 Consider the random variable $X \sim U(0,1)$ and the random variable $Y \sim U(0,1)$. Find the distribution of $V = XY$.

This is a simple application of the algorithm. The following Maple code defines the random variables X and Y and returns the PDF of their product. Note, the procedure `UniformRV` returns the PDF in a Maple list-of-lists data structure outlined in Glen, Leemis, and Drew (1997).

```
X := UniformRV(0, 1);
Y := UniformRV(0, 1);
V := Product(X, Y);
PDF(V);
```

The resulting PDF for $V = XY$ is

$$h(v) = -\ln v \qquad\qquad 0 < v < 1\,,$$

which is readily verified by hand.

Example 4.2 Consider the random variable $X \sim U(1,2)$ and the random variable $Y \sim U(3,4)$. Find the distribution of $V = XY$.

This is a straightforward application of the algorithm in that all segments of X and Y are in the first quadrant. In this example, the PDF is more tedious to calculate by hand. The program yields the following PDF for $V = XY$:

$$h(v) = \begin{cases} \ln v - \ln 3 & 3 < v < 4 \\ \ln 4 - \ln 3 & 4 < v < 6 \\ 3\ln 2 - \ln v & 6 < v < 8 \end{cases} .$$

Note that while the PDF of both X and Y are defined on single segments that have positive interval limits, the PDF of V is defined on three segments.

Example 4.3 Consider the random variable $X \sim U(-1, 2)$ and the random variable $Y \sim U(-3, 4)$. Find the PDF of $V = XY$. This example tests whether the algorithm handles the case of 0 in the support of X and Y correctly. The program yields the correct PDF for V:

$$h(v) = \begin{cases} \frac{1}{21}\ln\left(-\frac{6}{v}\right) & -6 < v < -4 \\ \frac{1}{21}\ln\left(\frac{24}{v^2}\right) & -4 < v < 0 \\ \frac{1}{21}\ln\left(\frac{24}{v^2}\right) & 0 < v < 3 \\ \frac{1}{21}\ln\left(\frac{8}{v}\right) & 3 < v < 8 \end{cases} .$$

Maple returns a mathematically equivalent PDF with a slightly different functional form for the second and third segments of V. The second segment, for example, is given by Maple as $\frac{1}{7}\ln(2) - \frac{1}{21}\ln(v) + \frac{1}{21}I\pi + +\frac{1}{21}\ln(3) - \frac{1}{21}\ln(-v)$, which reduces to the above second segment given above since $\ln(-1) = I\pi$, so the imaginary portions cancel.

Example 4.4 Consider the random variable $X \sim Triangular(1, 2, 3)$ and the random variable $Y \sim Triangular(1, 2, 4)$. Find the PDF of $V = XY$. This non-uniform example illustrates the case of several rectangular product spaces. The Maple code in this case is:

```
X := TriangularRV(1, 2, 3);
Y := TriangularRV(1, 2, 4);
```

```
V := Product(X, Y);
PDF(V);
```

The resulting PDF for V has six segments:

$$h(v) = \begin{cases} -4v/3 + 2/3\ln v + 2v/3\ln v + 4/3 & 1 < v < 2 \\ -8 + 14/3\ln 2 + 7v/3\ln 2 + 10v/3 - 4\ln v & \\ \quad -5v/3\ln v & 2 < v < 3 \\ -4 + 14/3\ln 2 + 7v/3\ln 2 + 2v - 2\ln v & \\ \quad -v\ln v - 2\ln 3 - 2v/3\ln 3 & 3 < v < 4 \\ 44/3 - 14\ln 2 - 7v/3\ln 2 - 8v/3 - 2\ln 3 & \\ \quad +22/3\ln v - 2v/3\ln 3 + 4v/3\ln v & 4 < v < 6 \\ 8/3 - 8\ln 2 - 4v/3\ln 2 - 2v/3 + 4/3\ln v & \\ \quad +v/3\ln v + 4\ln 3 + v/3\ln 3 & 6 < v < 8 \\ -8 + 8\ln 2 + 2v/3\ln 2 + 2v/3 + 4\ln 3 & \\ \quad -4\ln v + v/3\ln 3 - v/3\ln v & 8 < v < 12 \end{cases} .$$

Example 4.5 We give an illustration of the fact that the product of two lognormal random variables has the lognormal distribution. Consider the random variable $X \sim LogN(\mu_1, \sigma_1^2)$ and the random variable $Y \sim LogN(\mu_2, \sigma_2^2)$. Find the PDF of $V = XY$. To illustrate, we set $\mu_1 = \mu_2 = 0$, $\sigma_1^2 = 1$ and $\sigma_2^2 = 4$. The program yields the following PDF:

$$h(v) = \frac{e^{-(\ln v)^2/10}}{v\sqrt{10\pi}} \qquad 0 < v < \infty\,.$$

This can be identified as a $LogN(\mu = 0, \sigma^2 = 5)$ random variable.

Example 4.6 Consider the random variable $X \sim N(0,1)$ and the random variable $Y \sim N(0,1)$. Find the PDF of $V = XY$. This will test the case where the support of X and Y include the endpoints $\pm\infty$.

The program yields the following PDF for V:

$$h(v) = \begin{cases} \dfrac{\text{BesselK}(0,\, v \cdot \text{signum}(v))}{\pi} & -\infty < v < 0 \\ \dfrac{\text{BesselK}(0,\, v \cdot \text{signum}(v))}{\pi} & 0 < v < \infty \end{cases} ,$$

which relies on Maple's `BesselK` and `signum` functions. A plot of this function is given in Figure 3.

Example 4.7 Devroye (1996) gives

$$X \leftarrow m + [a + (b - a)U_1 - m] \max\{U_2, U_3\}$$

as a one-line algorithm for generating a $Triangular(a, m, b)$ variate, where U_1, U_2, and U_3 are independent and identically distributed $U(0, 1)$ random variables and the triangular distribution has minimum a, mode m, and maximum b. We may now use this relationship to generate the PDF of a random variable with the triangular distribution. Using the Maple procedure `Maximum`, also written by Glen, which returns the PDF of the maximum of two independent random variables and `Transform` for the linear transformations, we can determine the PDF with the following Maple commands.

```
a  := 1;
m  := 2;
b  := 3;
U1 := UniformRV(0, 1);
U2 := UniformRV(0, 1);
U3 := UniformRV(0, 1);
T1 := Transform(U1, [[x -> a + (b - a) * x - m],[- infinity, infinity]]);
T2 := Maximum(U2, U3);
T3 := Product(T1, T2);
X  := Transform(T3, [[x -> m + x],[- infinity, infinity]]);
```

The resulting PDF for X is:

$$f(x) = \begin{cases} x - 1 & 1 < x < 2 \\ 3 - x & 2 < v < 3 \end{cases} .$$

Example 4.8 Consider the independent random variables $U_1 \sim U(0, 1)$ and $U_2 \sim U(0, 1)$. The Box–Muller algorithm for generating a single standard normal

deviate V can be coded in one line (Devroye, 1996) as

$$V \leftarrow \sqrt{-2\ln U_1}\cos(2\pi U_2),$$

where U_1 and U_2 are independent random numbers. Using the `Transform` (Glen, Leemis, and Drew, 1997) and `Product` procedures together, one can determine the PDF of V. Due to the principle inverse difficulty with trigonometric functions, however, the transformation must be rewritten as

$$V \leftarrow \sqrt{-2\ln U_1}\cos(\pi U_2)$$

before using `Transform`.

The program yields the following PDF for V:

$$h(v) = \begin{cases} \dfrac{v}{\pi}\displaystyle\int_{-1}^{0} \frac{e^{-v^2/(2x^2)}}{\sqrt{1-x^2}\,x^2}\,dx & -\infty < v < 0 \\ \dfrac{v}{\pi}\displaystyle\int_{0}^{1} \frac{e^{-v^2/(2x^2)}}{\sqrt{1-x^2}\,x^2}\,dx & 0 < v < \infty \end{cases}.$$

While this form in not easily recognizable as the PDF for the normal distribution, it is mathematically equivalent to the more standard

$$h(v) = \frac{1}{\sqrt{2\pi}}e^{-v^2/2} \qquad -\infty < v < \infty.$$

We anticipate that future generations of computer algebra systems will be able to simplify these integrals.

As a final example, the `Product` procedure can be used in various types of statistical inference, as illustrated here in hypothesis testing.

Example 4.9 (Hogg and Craig, 1995, page 287) Let X_1 and X_2 be iid observations drawn from a population with PDF

$$f(x) = \theta x^{\theta-1} \qquad 0 < x < 1,$$

where $\theta > 0$. Test H_0: $\theta = 1$ versus H_1: $\theta > 1$ using the test statistic X_1X_2 and the critical region $C = \{(X_1, X_2)|X_1X_2 \geq 3/4\}$. Find the significance level α and power function for the test.

The APPL code to compute the power function is

```
n := 2;
crit := 3 / 4;
assume(theta > 0);
X := [[x -> theta * x ^ (theta - 1)], [0, 1], ["Continuous", "PDF"]];
T := ProductIID(X, n);
power := SF(T, crit);
```

which yields

$$\Pr(\text{rejecting } \ H_0|\theta) = 1 - (3/4)^\theta + \theta(3/4)^\theta \ln(3/4).$$

The fact that the population distribution is non-standard indicates that `X` must be defined using the list-of-three lists data structure shown above. The `assume` statment defines the parameter space. The `ProductIID` procedure simply makes repeated calls to `Product`. Finally, the `SF` procedure returns the survivor function, which is the complement of the CDF. To compute the significance level of the test, the additional Maple statement

```
subs(theta = 1, power);
```

is required, yielding $\alpha = 1/4 + 3/4 \ln(3/4) \cong 0.0342$. Maple's floating point evaluator `evalf` can be used to compute the floating point representation of α. To plot the power function requires the additional statement

```
plot(power, theta = 0 .. 4);
```

Obviously, this example can be generalized for different sample sizes, population distributions, and critical values with only minor modification.

5 Conclusion

An algorithm for calculating the PDF of the product of two independent random variables X and Y (which may be defined in a piece-wise manner) has been developed and implemented.

The APPL procedure `Product` is one of many procedures capable of automating complicated probability calculations associated with random variables (Glen, Evans, and Leemis, 2001). Potential application areas for calculations of this type occur in all areas of applied probability and statistics.

References

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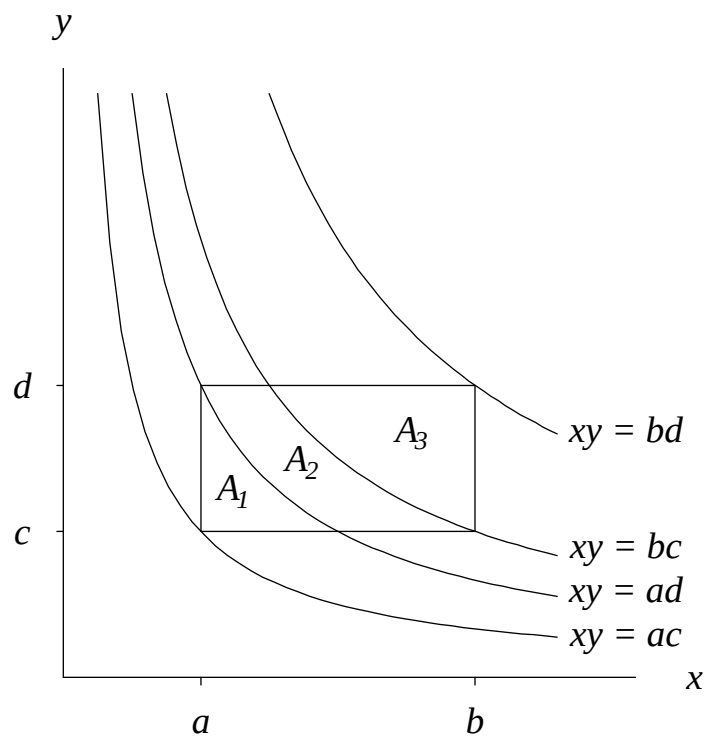

Figure 1: The support of X and Y when $ad < bc$.

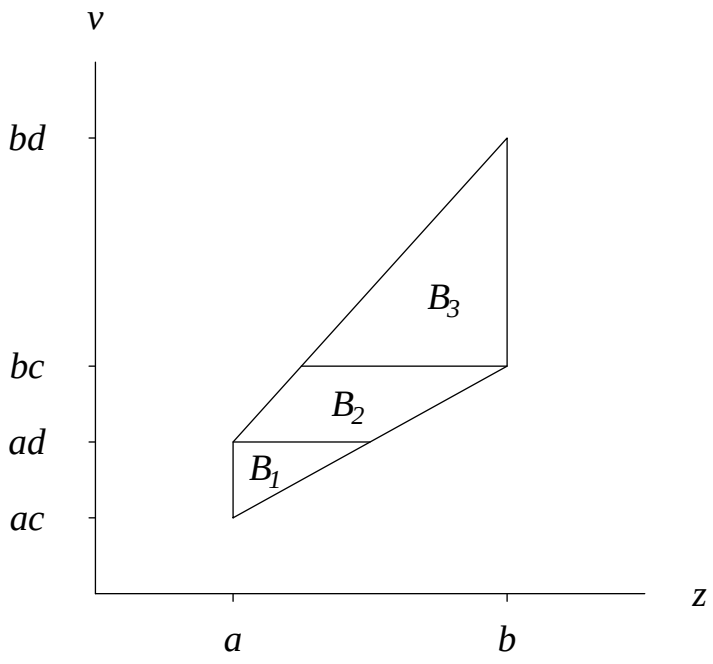

Figure 2: The mapping of $Z = X$ and $V = XY$ when $ad < bc$.

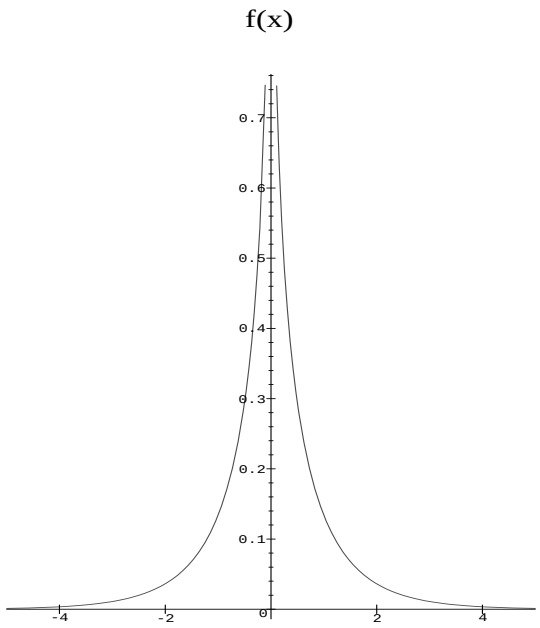

Figure 3: The PDF of $V = XY$ for $X \sim N(0,1)$ and $Y \sim N(0,1)$.

Procedure `Product`

`Input`: The pdf of X, $f(x)$, and the pdf of Y, $g(y)$, where X and Y are independent, continuous random variables. The pdf's are in the list-of-lists format; e.g., $[fX_1, fX_{2,}[\text{`}\texttt{Continuous}\text{`}, \text{`}\texttt{PDF}\text{`}]]$, as described in Glen, Leemis, and Drew (1997).

`Output`: The pdf of $V = X \cdot Y$, $h(v)$, in the list-of-lists format. The pdf is determined by the general result

$$h(v) = \int_{-\infty}^{\infty} f(x) \cdot g\left(\frac{v}{x}\right) \frac{1}{|x|} dx.$$

$X^\star \leftarrow fX_2$ (from the list-of-lists format), $n \leftarrow \|X^\star\|$

$Y^\star \leftarrow fY_2, m \leftarrow \|Y^\star\|$

$f \leftarrow fX_1$

$g \leftarrow fY_1$

$V^\star \leftarrow [\,]$, (the empty list)

$h \leftarrow [\,]$

Insert 0 into $X^\star$ if necessary

If ($X_1^\star < 0$ and $X_n^\star > 0$ and $0 \notin X^\star$) then

  For $i \leftarrow 1$ to n

    If ($X_i^\star < 0$ and $X_{i+1}^\star > 0$) then

      Insert 0 between positions $X_i^\star$ and $X_{i+1}^\star$

      Insert f_i at position f_{i+1}

      $n \leftarrow n + 1$

      break

Insert 0 into $Y^\star$ if necessary

If ($Y_1^\star < 0$ and $Y_m^\star > 0$ and $0 \notin Y^\star$) then

  For $i \leftarrow 1$ to m

    If ($Y_i^\star < 0$ and $Y_{i+1}^\star > 0$) then

      Insert 0 between positions $Y_i^\star$ and $Y_{i+1}^\star$

      Insert g_i at position g_{i+1}

      $m \leftarrow m + 1$

      break

Set up the support list $V^\star$

For $i \leftarrow 1$ to n

  For $j \leftarrow 1$ to m

    $V^\star \leftarrow V^\star \bigcup \left(X_i^\star \cdot Y_j^\star\right)$

$l \leftarrow \|V^\star\|$

For $i \leftarrow 1$ to $l-1$

  $h_i \leftarrow 0$

Integrate for each piecewise segment of X times each piecewise segment of Y

For $i \leftarrow 1$ to $n-1$

  For $j \leftarrow 1$ to $m-1$

    $a \leftarrow X_i^\star$

    $b \leftarrow X_{i+1}^\star$

    $c \leftarrow Y_j^\star$

    $d \leftarrow Y_{j+1}^\star$

    If ($a \geq 0$ and $c \geq 0$) then

      # The two segments transform into the 1st Quadrant

      $f1 \leftarrow \int_a^b f_i(x) \cdot g_j(\frac{v}{x}) \cdot \frac{1}{x} dx$

      If ($d < \infty$) then $f2 \leftarrow \int_{v/d}^b f_i(x) \cdot g_j(\frac{v}{x}) \cdot \frac{1}{x} dx$

      If ($c > 0$) then $f3 \leftarrow \int_a^{v/c} f_i(x) \cdot g_j(\frac{v}{x}) \cdot \frac{1}{x} dx$

      If ($c > 0$ and $d < \infty$ and $ad < bc$) then $f4 \leftarrow \int_{v/d}^{v/c} f_i(x) \cdot g_j(\frac{v}{x}) \cdot \frac{1}{x} dx$

      #1st Quad, Scenario A

      If ($c = 0$ and $d = \infty$) then

        For $ii \leftarrow 1$ to $l-1$

          If ($V_{ii}^\star \geq 0$ and $V_{ii+1}^\star \leq \infty$) then

            $h_{ii} \leftarrow h_{ii} + f1$

      #1st Quad, Scenario B

      If ($c = 0$ and $d < \infty$) then

        For $ii \leftarrow 1$ to $l-1$

          If ($V_{ii}^\star \geq 0$ and $V_{ii+1}^\star \leq ad$) then

$h_{ii} \leftarrow h_{ii} + f1$

If ($V^{\star}_{ii} \geq ad$ and $V^{\star}_{ii+1} \leq bd$) then

$h_{ii} \leftarrow h_{ii} + f2$

#1st Quad, Scenario C

If ($c > 0$ and $d = \infty$) then

For $ii \leftarrow 1$ to $l - 1$

If ($V^{\star}_{ii} \geq bc$ and $V^{\star}_{ii+1} \leq \infty$) then

$h_{ii} \leftarrow h_{ii} + f1$

If ($V^{\star}_{ii} \geq ac$ and $V^{\star}_{ii+1} \leq bc$) then

$h_{ii} \leftarrow h_{ii} + f3$

#1st Quad, Scenario D

If ($c > 0$ and $d < \infty$) then

If ($ad < bc$) then

#1st Case

For $ii \leftarrow 1$ to $l - 1$

If ($V^{\star}_{ii} \geq ac$ and $V^{\star}_{ii+1} \leq ad$) then

$h_{ii} \leftarrow h_{ii} + f3$

If ($V^{\star}_{ii} \geq ad$ and $V^{\star}_{ii+1} \leq bc$) then

$h_{ii} \leftarrow h_{ii} + f4$

If ($V^{\star}_{ii} \geq bc$ and $V^{\star}_{ii+1} \leq bd$) then

$h_{ii} \leftarrow h_{ii} + f2$

If ($ad = bc$) then

#2nd Case

For $ii \leftarrow 1$ to $l - 1$

If ($V^{\star}_{ii} \geq ac$ and $V^{\star}_{ii+1} \leq ad$) then

$h_{ii} \leftarrow h_{ii} + f3$

If ($V^{\star}_{ii} \geq bc$ and $V^{\star}_{ii+1} \leq bd$) then

$h_{ii} \leftarrow h_{ii} + f2$

If $(ad > bc)$ then

#3rd Case

For $ii \leftarrow 1$ to $l - 1$

If $(V^{\star}_{ii} \geq ac$ and $V^{\star}_{ii+1} \leq bc)$ then

$h_{ii} \leftarrow h_{ii} + f3$

If $(V^{\star}_{ii} \geq bc$ and $V^{\star}_{ii+1} \leq ad)$ then

$h_{ii} \leftarrow h_{ii} + f1$

If $(V^{\star}_{ii} \geq ad$ and $V^{\star}_{ii+1} \leq bd)$ then

$h_{ii} \leftarrow h_{ii} + f2$

Repeat the algorithm for quadrants II, III, and IV.

Remove any reduncancies in h due to 0 in the support list.

Make a new list of lists for V

$fV_1 \leftarrow h$

$fV_2 \leftarrow V^{\star}$

$fV_3 \leftarrow [`Continuous`, `PDF`]$

`Return`(fV)