Chapter 1

Introduction

Figure 1.1: The GPS track of a walk viewed in Google Earth

1.1 A First Problem

Welcome to **Modeling in a Real and Complex World.** This book is about using modeling to solve real and significant problems, like the looming energy crisis and global warming, whose solutions require the kind of understanding that is possible only by building and using sophisticated mathematical models.

Figures 1.1, and 1.2 show two powerful kinds of modeling we use frequently – maps and photographic imagery. These models work well together with mathematical models. In our first problem we begin by using these figures to get the big picture of a walk taken by one of the authors. Then we use mathematical models together with GPS[1] data to find some specific information – the length of the walk. You can download the Google Earth .kmz file shown in Figure 1.1[2] to your computer and then open it from within Google Earth to look at this walk using all the power of Google Earth.

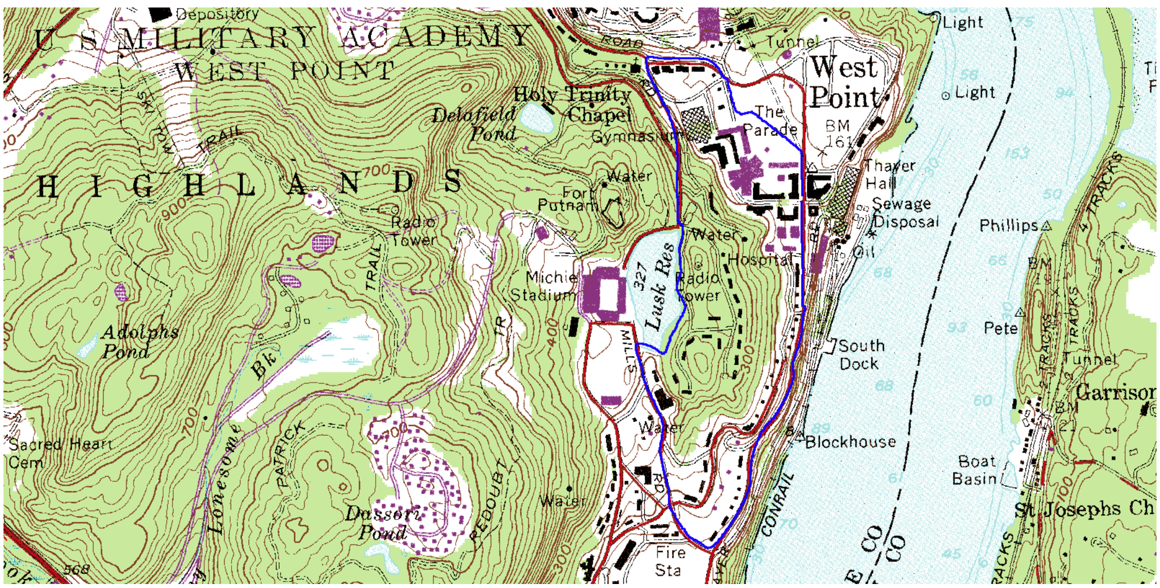

Figure 1.2: The GPS track of the same walk on a topographic map

Looking at Figures 1.1, and 1.2 you can get a rough idea of the distance covered by the walk. The walk followed a counterclockwise loop that started and ended at the point marked by the red dot in Figure 1.1. The first part of the walk was on flat terrain but then the walk ascended a hill up toward Lusk Reservoir. Later it descended and then continued

[1] Global Positioning System.

[2] http://www.dean.usma.edu/math/people/Heidenberg/JOMA_GPS/Class-Materials/Intro-modeling/walk.kmz

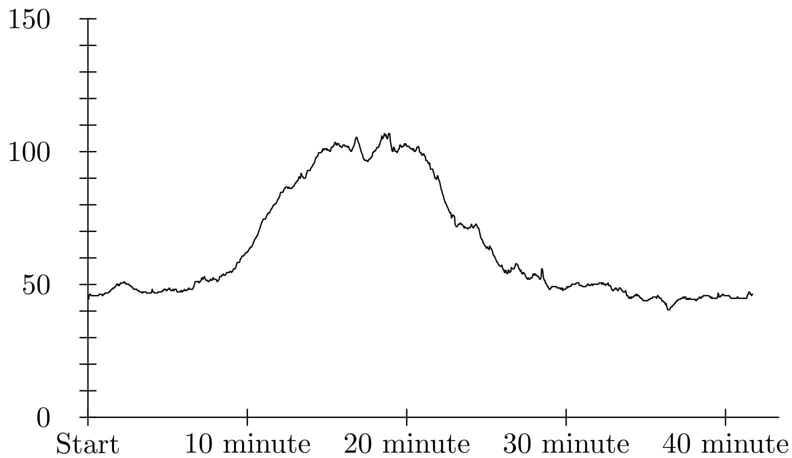

Figure 1.3: Elevation in meters

on fairly level terrain back to the starting point. Figure 1.3 shows the elevation in meters above sea level measured by the GPS unit during the walk. Notice all the little bumps. It looks like the author was doing some pretty dramatic jumping – he wasn't. Some of the bumps are due to noise in the data. You can match the broad variations in elevation shown in Figure 1.3 to the terrain shown in Figures 1.1 and 1.2. Table 1 shows a few lines of the raw data. The full data set can be downloaded here[3].

Date	Time	Latitude	Longitude	Elevation (meters)
12/15/06	10:01:22	41.39035	-73.95621	44.4
12/15/06	10:01:24	41.39035	-73.95623	45.3
12/15/06	10:01:26	41.39035	-73.95623	46.3
12/15/06	10:01:28	41.39035	-73.95623	46.3
12/15/06	10:01:30	41.39039	-73.95623	46.3
12/15/06	10:01:32	41.39041	-73.95623	46.3
12/15/06	10:01:34	41.39043	-73.95623	45.8
12/15/06	10:01:36	41.39048	-73.95623	45.8
12/15/06	10:01:38	41.39050	-73.95623	45.8

Table 1: A few lines of GPS data from the same walk

One of the difficulties of working with real data is that the noise that can come from measurement error or from unusual events can obscure broad trends. These problems are particularly important as we look at two of the most important problems facing the world – global warming and limited energy. A few particularly hot or cold days in one part of the world are not evidence by themselves for or against global warming and the dramatic

[3]http://www.dean.usma.edu/math/people/Heidenberg/JOMA_GPS/Class-Materials/Intro-modeling/walk.xls

gasoline price jump after Hurricane Katrina doesn't tell us much about the long term trends in supply and demand for oil. It can be quite difficult to decide what is noise and what is not. GPS units, for example, can be affected by large buildings.

We are now ready to tackle our first problem – we want to determine the length of this walk. In your class you will work with data that you collect. Your data may look at a walk, a run, a cross-country ski trip, a bicycle trip, a car trip, or even a helicopter flight.

The first difficulty we encounter is the units in which our data is given. The most important data – geographical location – is given in terms of latitude and longitude. These units are not the usual units in which we discuss length. We want to work with units like feet, miles, meters, or kilometers. The problem of units is very common. For example, if you are planning a trip to Europe you need to know how to convert prices in euros to prices in U.S. dollars. Converting locations expressed in latitude and longitude into locations expressed in kilometers is more difficult than converting euros to U.S. dollars[4]. In particular, it requires us to have a mental image or model of the earth.

The earth is shaped like a slightly squashed sphere. Its polar radius is roughly 6,357 kilometers and its equatorial radius is roughly 6,378 kilometers. The difference between the two is about 21 kilometers, or 0.3%. Our model of the earth will be a sphere of radius 6,367.5 kilometers, or 6,367,500 meters. This model will be good enough for our purposes.

Suppose that two points have the same longitude but that their latitude is different. Suppose that the latitude of one point is y_1 and the latitude of the second point is y_2. Both y_1 and y_2 are expressed in degrees with decimals, like the entries in Table 1. It is much easier to work with degrees in this form than with degrees, minutes, and seconds. The distance measured in degrees between these two points is just $|y_2 - y_1|$. Since the radius of the earth is 6,367,500 meters we can convert this distance to meters using the formula

$$|y_2 - y_1| \times \frac{2 \times \pi \times 6,367,500}{360}$$

or, roughly,

$$111,134 \times |y_2 - y_1|.$$

Next we look at two points with the same latitude but different longitudes, x_1 and x_2. The same formula would work if the two points were on the equator but most of us do not live on the equator. Circles going around the earth at latitudes close to either of the poles are much smaller than circles going around the earth at latitudes close to the equator. Using Figure 1.4 we see that the radius, S, of the circle around the earth at latitude θ is

$$S = 6,367,500 \cos\theta.$$

[4] Actually converting dollars to euros has some difficulties too. The conversion factors depend on time. For example, on 1 July 2001 the euro was worth \$0.83 and on 16 December 2006 it was worth \$1.32. The conversion factors for physical units like meters and feet do not depend on date or time.

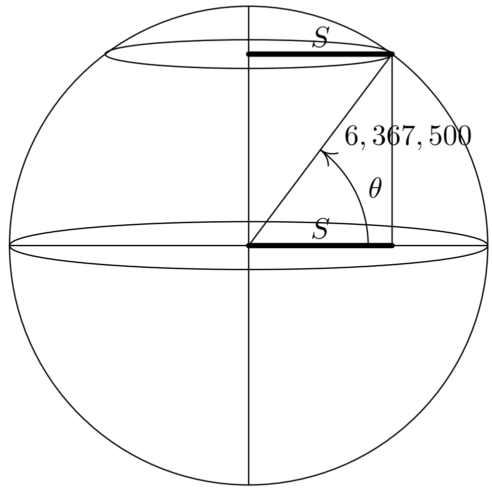

Figure 1.4: Circles at different latitudes

Before using the cosine function on the latitude we must convert latitude expressed in degrees to latitude expressed in radians. This give us the formula

$$|x_2 - x_1| \times \frac{2 \times \pi \times 6,367,500 \times \cos\left(\frac{2\pi y}{360}\right)}{360}$$

or, roughly

$$111,134 \times |x_2 - x_1| \times \cos\left(\frac{2\pi y}{360}\right)$$

where y is the latitude in degrees of the two points. We can summarize our two formulas using the notation Δx for the east-west distance in meters and Δy for the north-south distance in meters by

$$\begin{aligned} \Delta x &= 111,134 \times |x_2 - x_1| \times \cos\left(\frac{2\pi\theta}{360}\right) \\ \Delta y &= 111,134 \times |y_2 - y_1|. \end{aligned}$$

Unless you're a pilot or an astronaut, you probably live your day-to-day life as if the world were flat. For example, if you walk west for 1.53 hours at 4 miles per hour you can be quite confident that you will be 6.12 miles from your starting point and the fastest way back is to walk 6.12 miles east. The International Space Station, however, is flying so fast

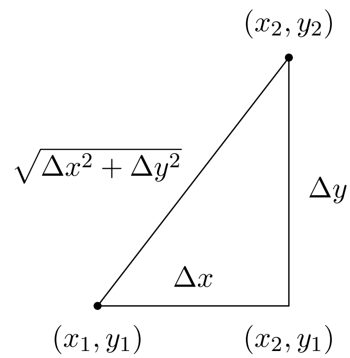

Figure 1.5: The distance between two points

that we cannot ignore the fact that the earth is not flat. After flying in the same direction for 1.53 hours it is right back where it started.

For our walk, we can work with a flat model for our part of the earth. Using Figure 1.5 and the Pythagorean Theorem we see that the distance between two points (x_1, y_1) and (x_2, y_2), using latitude-longitude coordinates in decimal degrees, is approximately[5]

$$\sqrt{\Delta x^2 + \Delta y^2} = 111,134\sqrt{(x_2 - x_1)^2 \cos^2\left(\frac{2\pi y_1}{360}\right) + (y_2 - y_1)^2}.$$

Note that x_1 and x_2 are the longitude coordinates and that y_1 and y_2 are the latitude coordinates.

Example 1 *Washington Hall is located at longitude*[6] *73.95799 W, latitude 41.39143 N. Thayer Hall is located at longitude 73.95941 W, latitude 41.39030 N. Find the distance between the two.*

Answer: *The solution is a straightforward calculation – roughly 318 meters.*

Example 2 *The Empire State Building is located at longitude 73.98 W and latitude 40.75 N. Find the distance between Washington Hall and the Empire State Building. Then find the distance between the Empire State Building and Washington Hall. Be very, very careful. This is not a misprint. The two distances are different. Why are they different? Why is the difference so small?*

[5]We must use the qualifying adverb "approximately" because this formula relies on a less than perfect model.

[6]We denote longitude west of the prime meridian through Greenwich England by negative numbers or by "W". Similarly we denote latitude south of the equator by negative numbers of "S."

Answer: *Straightforward calculations using the formula*

$$111,134\sqrt{(x_2-x_1)^2\cos^2\left(\frac{2\pi y_1}{360}\right)+(y_2-y_1)^2}$$

very carefully (notice that y_1 appears twice in the formula but y_2 appears only once) shows that

- *the distance from Washington Hall to the Empire State Building is 71.3139 kilometers and*
- *the distance from the Empire State Building to Washington Hall is 71.3143 kilometers.*

The two answers are different but the difference is less than half a meter. This small difference is due to the fact that the earth is not really flat. This difference is small because our small part of the earth is pretty close to flat.

Figure 1.6: Four buildings in Washington, DC viewed in Google Earth

Question 1 *Using Google Earth the latitude and longitude of four buildings are*

- *The Washington Monument: 38.889448 N. 77.035204 W.*

- *The United States Capitol: 38.889420 N. 77.008913 W.*
- *The Jefferson Memorial: 38.881347 N. 77.036587 W.*
- *The Lincoln Memorial: 38.889237 N. 77.050118 W.*

Find the distance between each of the buildings.

Figure 1.7: Three cities in the United States viewed in Google Earth

Question 2 *Using Google Earth the latitude and longitude of three cities in the United States are*

- *San Francisco: 37.775000 N. 122.418333 W.*

- *New York City: 40.714167 N. 74.006389 W.*
- *Washington DC: 38.895000 N. 77.036667 W.*

Find the distance between each of the cities.

Question 3 *Suppose you were to fly in a straight line from San Francisco to Washington DC, then in a straight line from Washington DC to New York City, and then in a straight line from New York City back to San Francisco. How far would you travel? Discuss the accuracy of your estimate.*

Question 4 *Estimate the length of our sample walk. The data is here[7]. Find the best estimate you can and discuss how accurate you think your estimate is. What steps did you take to make it as accurate as possible? What potential sources of sources of inaccuracy did you consider?*

Question 5 *Collect your own GPS data for a walk, car trip, bicycle ride, or other trip. Estimate the length of your trip. Find the best estimate you can and discuss how accurate you think your estimate is. What steps did you take to make it as accurate as possible? What potential sources of sources of inaccuracy did you consider?*

[7] http://www.dean.usma.edu/math/people/Heidenberg/JOMA_GPS/Class-Materials/Intro-modeling/walk.xls