

# Signed Graph Coloring, the Art of Linear Algebra, and a Theorem of Jacobi

Steven Morse and Elisha Peterson

United States Military Academy

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# The famous 'function machine'

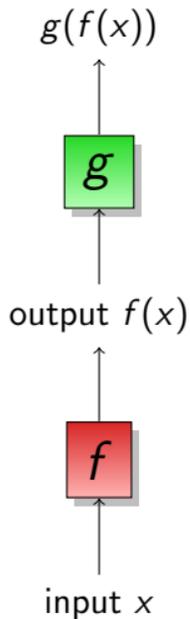
output  $f(x)$



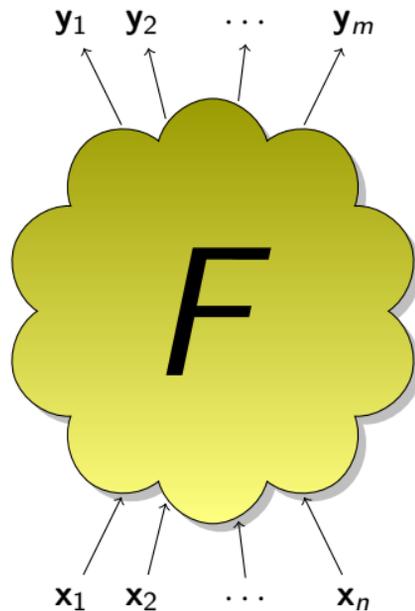
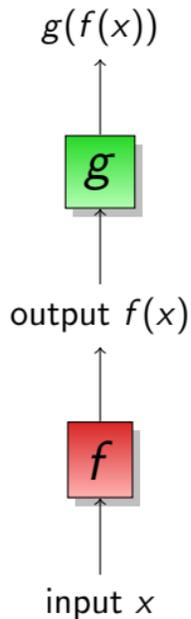
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# Signed Graph Coloring

## Definition of trace diagram

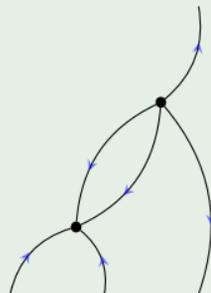
### Definition

An  $n$ -trace diagram is an oriented graph such that:

- vertices have degree  $n$  (if not leaves)
- all vertices sources or sinks
- edges at a vertex are ordered

How to define a trace diagram  
*signature*:

### Example



$$\text{sgn}(1\ 3\ 4\ 2) \cdot \text{sgn}(2\ 4\ 1\ 3) = -1$$

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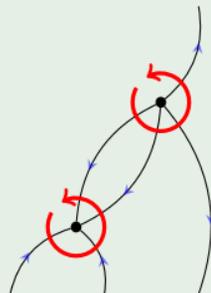
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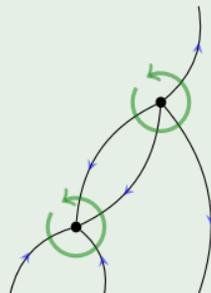
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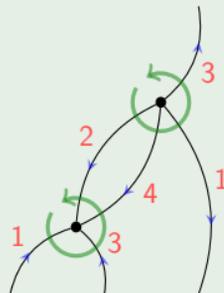
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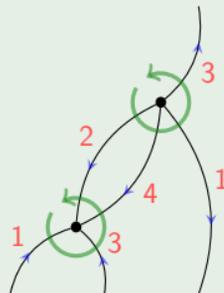
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# Associating a Function to a Diagram

- 1 Divide leaves into inputs and outputs
- 2 Mark input leaves (at bottom) by  $n$ -dimensional basis vectors
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## Example

What is the function ? Consider the input  $(b_1, b_2) = (1, 1)$

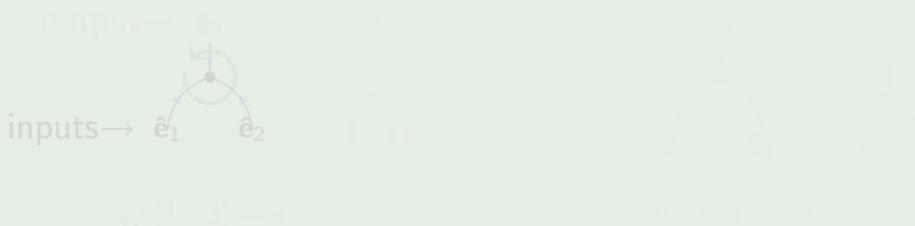


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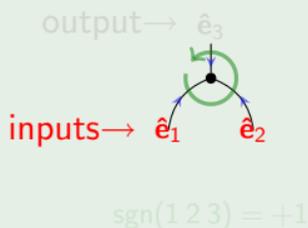
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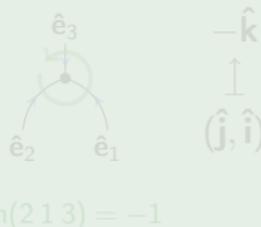
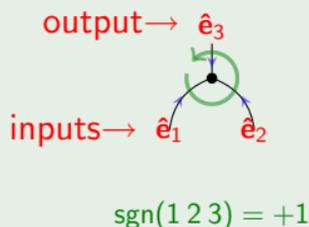
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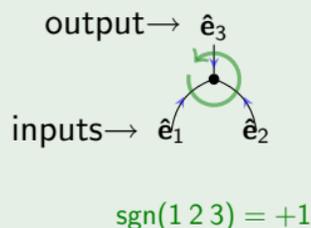
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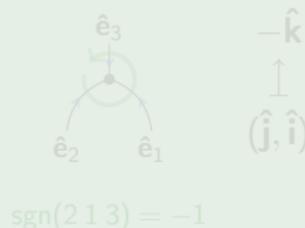
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$\hat{k}$

$\uparrow$

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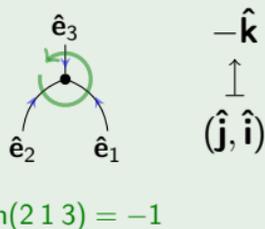
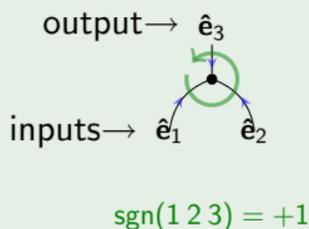
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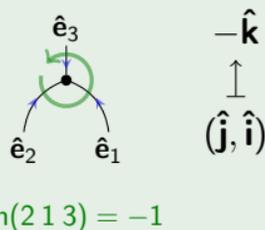
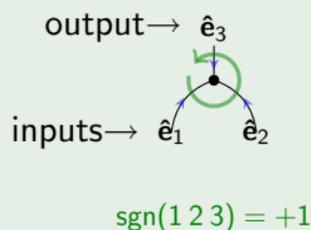
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# Marking diagrams with matrices

Matrix markings denote matrix multiplication:

$$\begin{array}{c} j \\ \uparrow \\ \textcircled{A} \\ \downarrow \\ i \end{array} = \hat{e}^j A \hat{e}_i = \begin{array}{c} \hat{e}_j \\ \uparrow \\ A \hat{e}_i \end{array} = a_{ij}$$

## Example



• There are no vertices, hence  $n$  colorings

• If the edge is colored by  $\pm 1$ , then the coloring is a  $\pm 1$ -eigenvector of  $A$ .

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$$\begin{array}{c} \textcircled{A} \\ \circlearrowleft \end{array} = \sum_i \begin{array}{c} i \\ \uparrow \\ \textcircled{A} \\ \downarrow \\ i \end{array} = \text{tr}(A)$$

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# The Art of Linear Algebra

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# How do you generalize the trace and the determinant??

Both are *invariants* that “output” a scalar:

$$\text{tr}(ABA^{-1}) = \text{tr}(B) \quad \det(ABA^{-1}) = \det(B)$$

What are the simplest “closed” diagrams?



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*Answer: These are the coefficients of the characteristic polynomial.*

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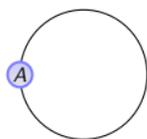
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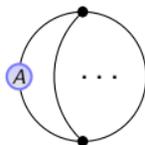
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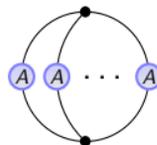
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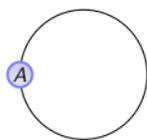
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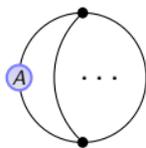
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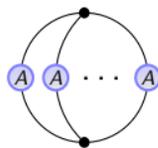
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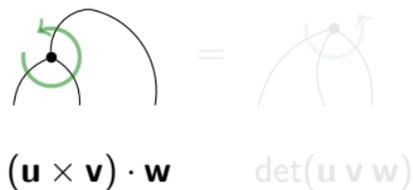
The diagram shows an equality between two expressions. On the left, a black dot is surrounded by three green curved arrows forming a cycle, representing the cross product  $\mathbf{u} \times \mathbf{v}$ . This is followed by a dot product with  $\mathbf{w}$ . On the right, a grey dot is surrounded by three grey curved arrows forming a cycle, representing the determinant of the matrix with columns  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ .

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \det(\mathbf{u} \ \mathbf{v} \ \mathbf{w})$$

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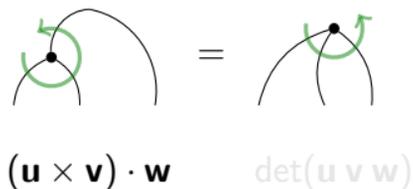
The diagram shows an equality between two expressions. On the left, a black dot is surrounded by three green arrows forming a circular pattern, representing the cross product  $\mathbf{u} \times \mathbf{v}$ . A black arrow points from this dot towards a black dot representing the vector  $\mathbf{w}$ . Below this is the expression  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ . On the right, three black arrows originate from a central black dot, representing the vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ . Below this is the expression  $\det(\mathbf{u} \ \mathbf{v} \ \mathbf{w})$ . An equals sign is placed between the two diagrams.

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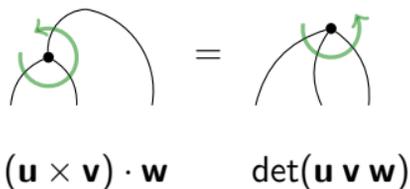
The diagram shows an equality between two expressions. On the left, a black dot is surrounded by three curved lines representing vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ . A green circular arrow indicates a counter-clockwise rotation from  $\mathbf{u}$  to  $\mathbf{v}$ . Below this is the expression  $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ . On the right, the same three vectors are shown, but the green arrow indicates a clockwise rotation from  $\mathbf{v}$  to  $\mathbf{u}$ . Below this is the expression  $\det(\mathbf{u} \ \mathbf{v} \ \mathbf{w})$ . An equals sign is placed between the two diagrams.

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# A Theorem of Jacobi

## Jacobi's theorem for matrix minors

## Theorem (Jacobi)

$$[\text{adj}(A)]_{I,J} = [A]^{m-1} \cdot [A]_{I^c, J^c}$$

- $A$  is a matrix,  $[A] \equiv \det(A)$
- $[A]_{I,J}$  is the minor (determinant of the submatrix formed from the set of  $m$  rows  $I$  and the set of  $m$  columns  $J$ )
- $\text{adj}(A)$  is the adjugate (matrix of  $(n-1) \times (n-1)$  minors)

## Example

$I = J = \{1, 2\}$ ,  $m = 2$ :

$$\begin{vmatrix} 2 & 1 & -1 & -4 \\ 1 & -2 & -1 & -6 \\ -1 & -1 & 2 & 4 \\ 2 & 1 & -3 & -8 \end{vmatrix}^{m-1}$$

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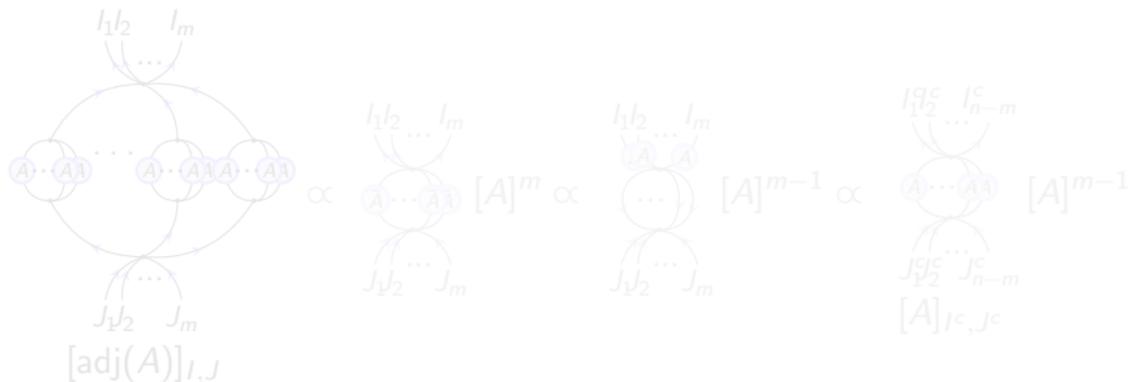
## Example

$I = J = \{1, 2\}$ ,  $m = 2$ :

$$\begin{vmatrix} -2 & -1 & -6 & & \\ -1 & 2 & 4 & \cdots & \\ 1 & -3 & -8 & & \\ & \cdots & & \cdots & \end{vmatrix} = \begin{vmatrix} 2 & 1 & -1 & -4 \\ 1 & -2 & -1 & -6 \\ -1 & -1 & 2 & 4 \\ 2 & 1 & -3 & -8 \end{vmatrix}^{m-1} \times \begin{vmatrix} 2 & 4 \\ -3 & -8 \end{vmatrix}$$

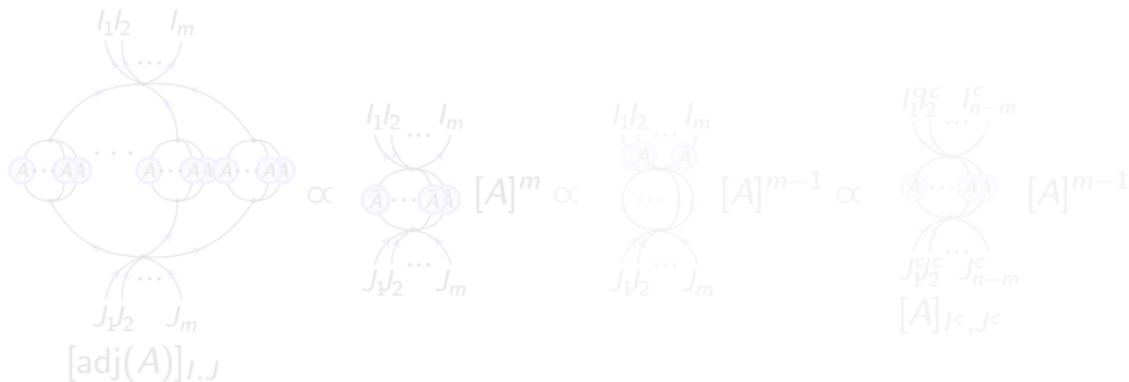
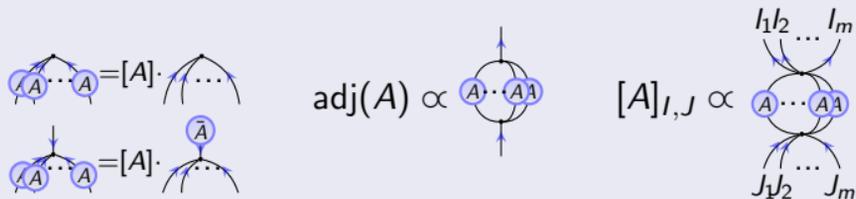
# One-Line Proof of $[\text{adj}(A)]_{I,J} = [A]^{m-1} \cdot [A]_{I^c, J^c}$ (Morse)

## Diagram Rules



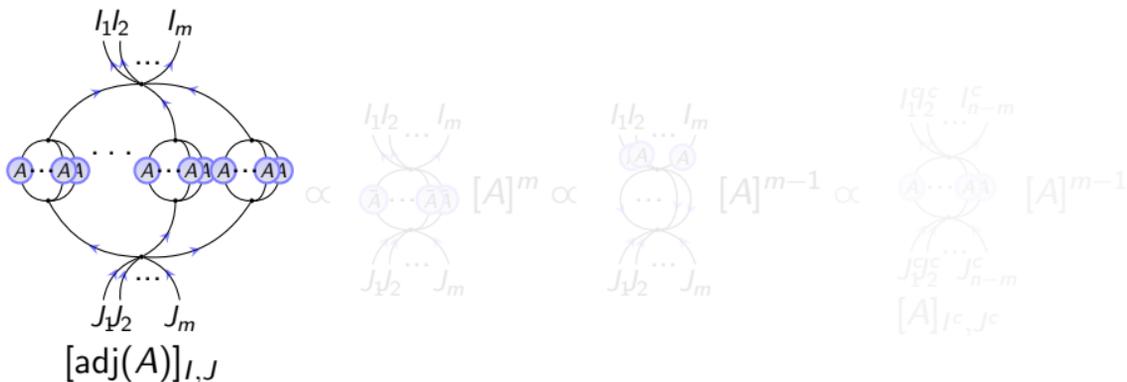
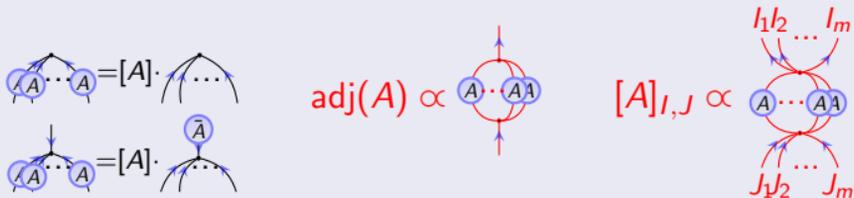
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## Diagram Rules



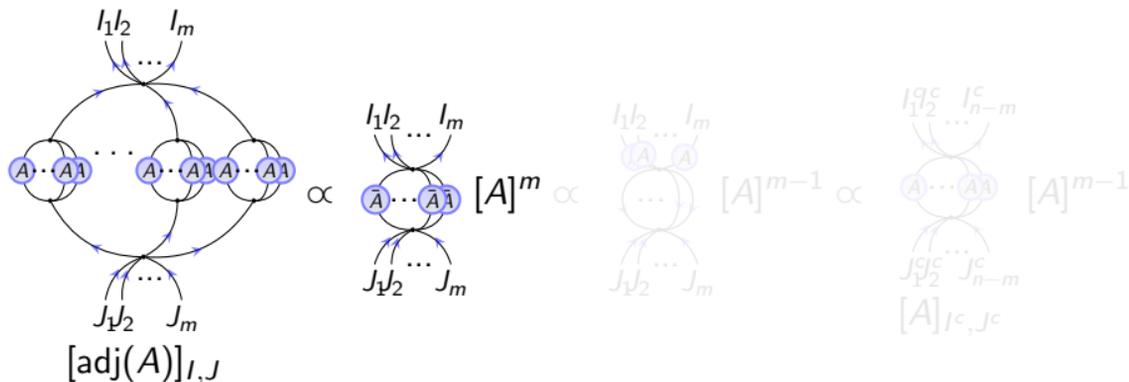
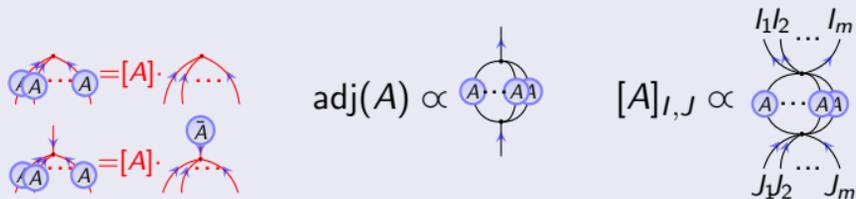
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## Diagram Rules



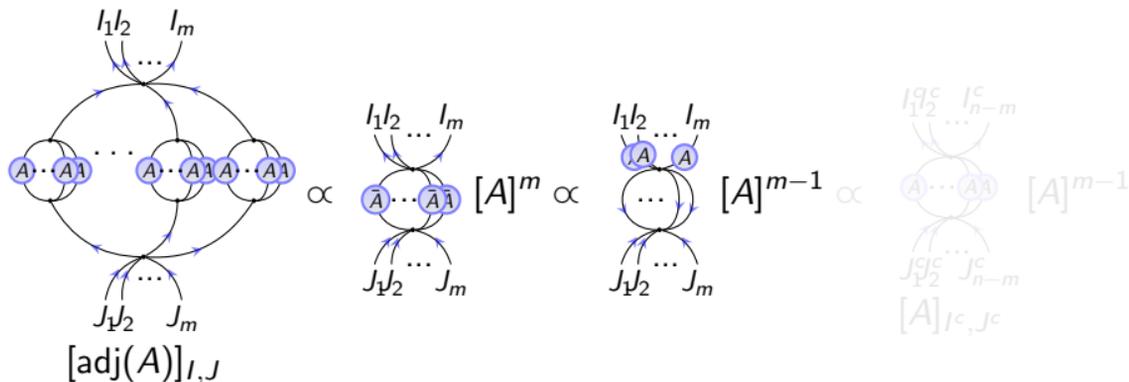
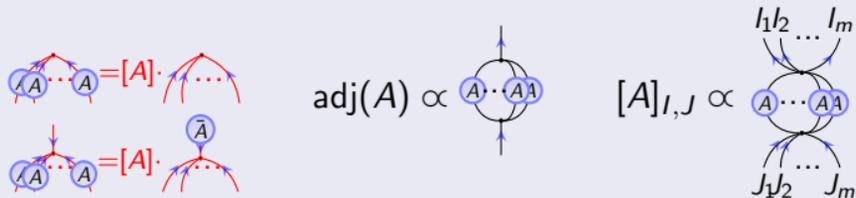
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## Diagram Rules



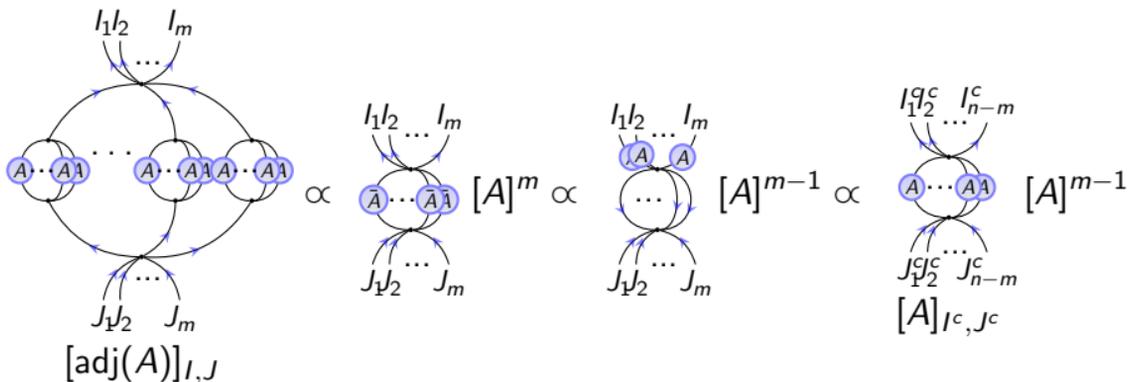
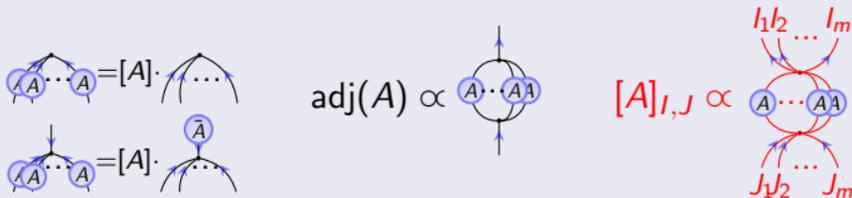
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## Diagram Rules



# One-Line Proof of $[\text{adj}(A)]_{I,J} = [A]^{m-1} \cdot [A]_{I^c, J^c}$ (Morse)

## Diagram Rules



## References

- Stedman, *Group Theory*
- Predrag Cvitanovic, *Group Theory*,  
<http://chaosbook.org/GroupTheory/>