Signed Graph Coloring, the Art of Linear Algebra, and a Theorem of Jacobi

Steven Morse and Elisha Peterson

United States Military Academy

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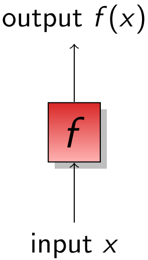
The famous 'function machine'



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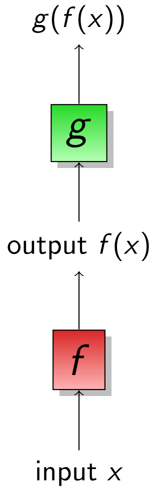

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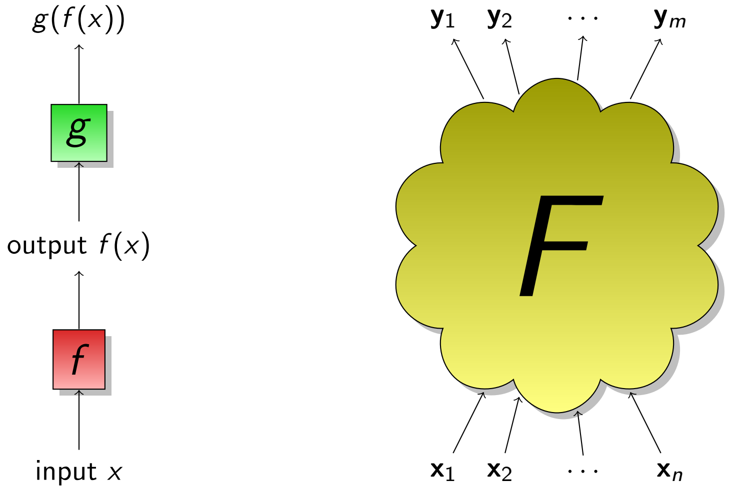


Signed Graph Coloring

Definition of trace diagram

Definition

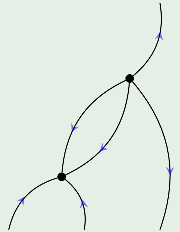
An *n-trace diagram* is an oriented graph such that:

- vertices have degree n (if not leaves)
- all vertices sources or sinks
- edges at a vertex are ordered

How to define a trace diagram *signature*:

1. Color the edges by n colors so that each color at a vertex is different
2. Multiply signatures of permutations at nodes

Example



$$\mathrm{sgn}(1\,3\,4\,2) \cdot \mathrm{sgn}(2\,4\,1\,3) = -1$$

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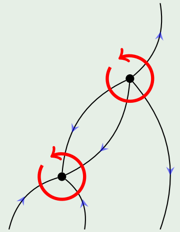

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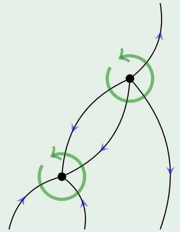

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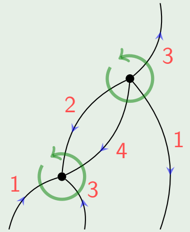

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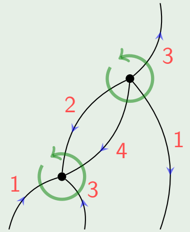

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Associating a Function to a Diagram

1. Divide leaves into inputs and outputs
2. Mark input leaves (at bottom) by n-dimensional basis vectors
3. Sum over labels of output leaves (at top) for all admissible colorings

Example

What is the function ? Consider the input $(\hat{e}_1, \hat{e}_2) = (\hat{i}, \hat{j})$:

It's the cross product!!

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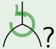
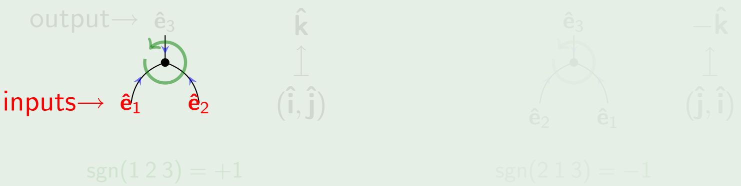
What is the function ? Consider the input $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2) = (\hat{\mathbf{i}}, \hat{\mathbf{j}})$:



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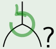
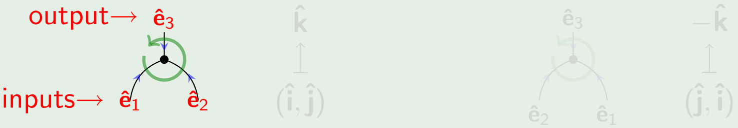
What is the function ? Consider the input $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2) = (\hat{\mathbf{i}}, \hat{\mathbf{j}})$:



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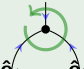

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$\hat{\mathbf{e}}_3$

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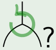
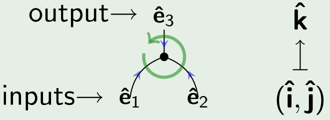
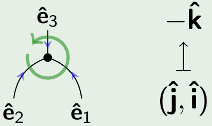
What is the function ? Consider the input $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2) = (\hat{\mathbf{i}}, \hat{\mathbf{j}})$:





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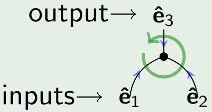

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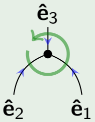

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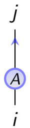
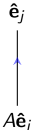
Marking diagrams with matrices

Matrix markings denote matrix multiplication:



$$= \hat{\mathbf{e}}^j A \hat{\mathbf{e}}_i =$$



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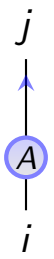

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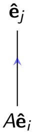

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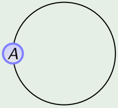

$= \sum_i$

$= \mathrm{tr}(A)$

- There are no vertices, hence n colorings
- If the edge is colored by i, the diagram is a_{ii}
- Sum over all colorings: $a_{11} + \cdots + a_{nn} = \mathrm{tr}(A)$

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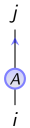

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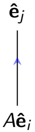

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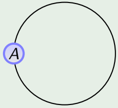

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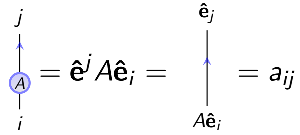

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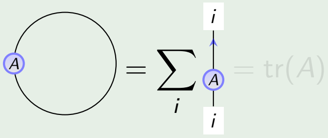

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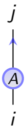

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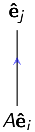

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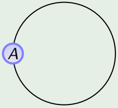

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The Art of Linear Algebra

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How do you generalize the diagram?

Answer: *The cross product generalizes to "products" of $n-1$ vectors*

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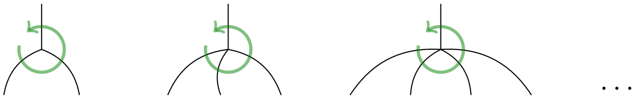

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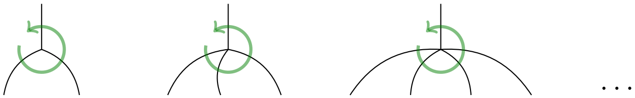

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How do you generalize the trace and the determinant??

Both are *invariants* that "output" a scalar:

$$\mathrm{tr}(ABA^{-1}) = \mathrm{tr}(B) \qquad \det(ABA^{-1}) = \det(B)$$

What are the simplest "closed" diagrams?

$\mathrm{tr}(A)$ $\quad \propto \mathrm{tr}(A)$ $\quad n-2$ more $\quad \propto \det(A)$

Answer: *These are the coefficients of the characteristic polynomial.*

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Can you prove anything with diagrams?

How are the determinant and cross product related?

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \qquad\qquad \det(\mathbf{u}\ \mathbf{v}\ \mathbf{w})$$

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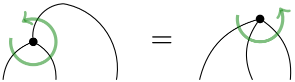

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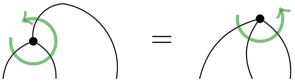

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A Theorem of Jacobi

Jacobi's theorem for matrix minors

Theorem (Jacobi)

$$[\mathrm{adj}(A)]_{I,J} = [A]^{m-1} \cdot [A]_{I^c,J^c}$$

- A is a matrix, $[A] \equiv \det(A)$
- $[A]_{I,J}$ is the minor (determinant of the submatrix formed from the set of m rows I and the set of m columns J)
- $\mathrm{adj}(A)$ is the adjugate (matrix of $(n-1) \times (n-1)$ minors)

Example

$I = J = \{1, 2\}$, $m = 2$:

$$\begin{vmatrix} 2 & 1 & -1 & -4 \\ 1 & -2 & -1 & -6 \\ -1 & -1 & 2 & 4 \\ 2 & 1 & -3 & -8 \end{vmatrix}^{m-1}$$

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$$[\operatorname{adj}(A)]_{I,J} = [A]^{m-1} \cdot [A]_{I^c,J^c}$$

- A is a matrix, $[A] \equiv \det(A)$
- $[A]_{I,J}$ is the minor (determinant of the submatrix formed from the set of m rows I and the set of m columns J)
- $\operatorname{adj}(A)$ is the adjugate (matrix of $(n-1) \times (n-1)$ minors)

Example

$I = J = \{1, 2\}$, $m = 2$:

$$[adj(A)]_{I,J}$$

$$\left|\begin{matrix} \left|\begin{matrix} -2 & -1 & -6 \\ -1 & 2 & 4 \\ 1 & -3 & -8 \end{matrix}\right| & |\cdots| \\ |\cdots| & |\cdots| \end{matrix}\right| = \left|\begin{matrix} 2 & 1 & -1 & -4 \\ 1 & -2 & -1 & -6 \\ -1 & -1 & 2 & 4 \\ 2 & 1 & -3 & -8 \end{matrix}\right|^{m-1} \times \left|\begin{matrix} 2 & 4 \\ -3 & -8 \end{matrix}\right|$$

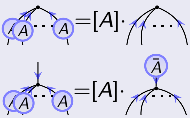
$$[A]_{I^c,J^c}$$

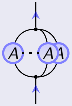
One-Line Proof of $[\mathrm{adj}(A)]_{I,J} = [A]^{m-1} \cdot [A]_{I^c,J^c}$ (Morse)

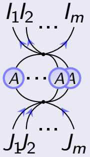
Diagram Rules

One-Line Proof of $[\mathrm{adj}(A)]_{I,J} = [A]^{m-1} \cdot [A]_{I^c,J^c}$ (Morse)

Diagram Rules



$\mathrm{adj}(A) \propto$ 

$[A]_{I,J} \propto$ 

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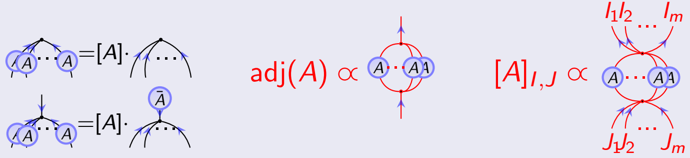

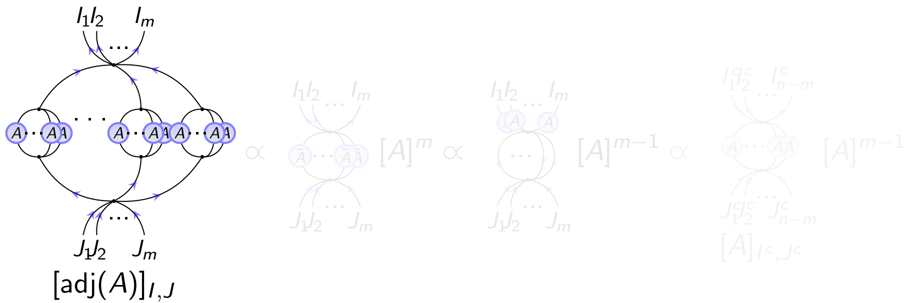

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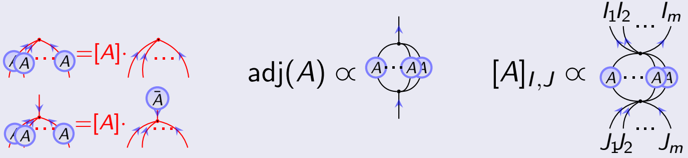

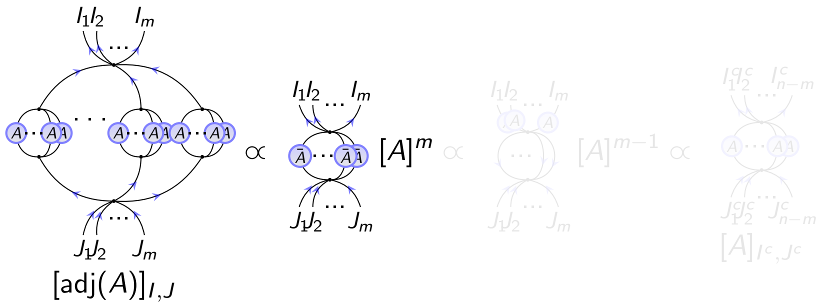

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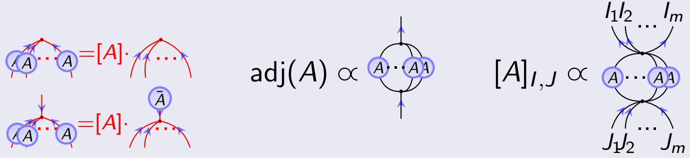

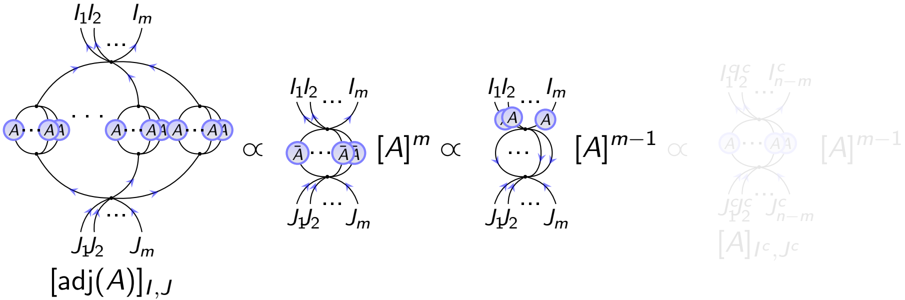

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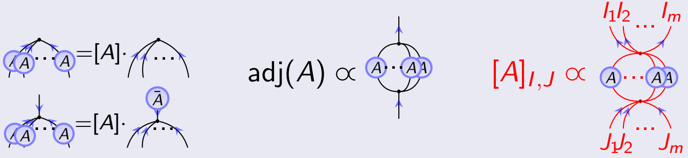

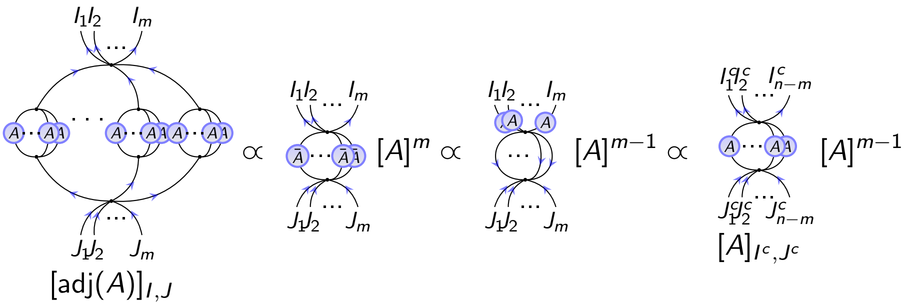

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- Predrag Cvitanovic, *Group Theory*, http://chaosbook.org/GroupTheory/