Trace Diagram Recurrences and Central Functions of $\mathrm{SL}(2, \mathbb{C})$-Character Varieties

AMS Special Session on Geometry, Algebra, and Topology of Character Varieties

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Character Varieties and their Coordinate Rings

Definition

The coordinate ring of a character variety $X = \mathrm{Hom}(\pi, G)//G$ is the ring of invariants $\mathbb{C}[\mathrm{Hom}(\pi, G)]^G$.

This talk concerns a certain basis of functions for $\mathbb{C}[\mathfrak{X}]$ when $G = \mathrm{SL}(2, \mathbb{C})$ and π is a free group, in which case $\mathbb{C}[\mathfrak{X}]$ is comprised of regular (polynomial) functions that satisfy

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Examples for $G = SL(2, C)$

π free of rank 1:

- $\mathbb{C}[\mathfrak{X}] \cong \mathbb{C}[x]$ and so can be written in terms of polynomials of one variable
- $\{1, x, x^2, x^3, \ldots\}$ comprises an additive basis

π free of rank 2:

- $\mathbb{C}[\mathfrak{X}] \cong \mathbb{C}[x, y, z]$ and so can be written in terms of polynomials of three variables (Fricke,Klein,Vogt)
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Computing an Additive Basis using the Peter-Weyl Theorem

Theorem (Corollary of Peter-Weyl)

For a reductive linear algebraic group G, the coordinate ring $\mathbb{C}[G]$ decomposes:

$$\mathbb{C}[G] \cong \bigoplus_{\lambda \in \Lambda} V_\lambda^* \otimes V_\lambda,$$

where Λ is the set of irreducible representations (of the maximal compact subgroup $U \subset G$), and the isomorphism is given by

$$v^* \otimes w \mapsto (x \mapsto v^*(x \cdot w)).$$

For $G = \mathrm{SL}(2, \mathbb{C})$, the irreducible representations are $V_0, V_1, V_2, \ldots$, the symmetric powers of the standard representation.

The Rank One Case

If π is free of rank 1, $\mathrm{Hom}(\pi, G) \cong G$ and so $\mathbb{C}[\mathfrak{X}] \cong \mathbb{C}[G]^G$.

By Schur's Lemma, $(V_n^* \otimes V_n)^G$ has dimension 1, so

$$\mathbb{C}[G]^G \cong \bigoplus_{n \in \mathbb{N}} (V_n^* \otimes V_n)^G \cong \bigoplus_n \mathbb{C}\chi_n$$

For a basis $\{v_i\}$ of V_n:

$$v^* \otimes w \mapsto \mathrm{tr}(x \mapsto v^*(x \cdot w)) = \sum_i v_i^*(x \cdot v_i).$$

In terms of the trace $\mathrm{tr}(X)$, the functions χ_n are the *Chebyshev polynomials of the second kind*:

$$\chi_0 = 1, \qquad \chi_n = x \cdot \chi_{n-1} - \chi_{n-2}.$$

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Definition of Central Functions

The following is a consequence of the *Peter-Weyl Theorem*:

Theorem (Central Function Decomposition)

The coordinate ring of the character variety $\mathbb{C}[\mathfrak{X}]$ can be written

$$\mathbb{C}[\mathfrak{X}] \cong \bigoplus_{\vec{\lambda},\psi,\phi} \mathbb{C}\chi_{\vec{\lambda}}^{\psi,\phi},$$

over particular injections ψ, ϕ into a tensor product of representations $V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$.

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Proof of the Central Function Decomposition

Proof.

When a surface Σ has fundamental group *free* of rank r, the isomorphism $\mathbb{C}[Hom(\pi, G)] \cong \mathbb{C}[G^{\times r}] \cong \mathbb{C}[G]^{\otimes r}$ and the previous result give:

$$\mathbb{C}[\mathfrak{X}] \cong \left(\mathbb{C}[G]^{\otimes r}\right)^G \cong \left(\bigotimes_r \bigoplus_{\lambda \in \Lambda} V_\lambda^* \otimes V_\lambda\right)^G$$
$$\cong \bigoplus_{(\lambda_1,\ldots,\lambda_r)\in\Lambda^r} \left(\left(V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_r}^*\right) \otimes \left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}\right)\right)^G.$$

Schur's Lemma and G-invariance permit a reduction to the desired form:

$$\mathbb{C}[\mathfrak{X}_r] \cong \bigoplus_{\vec{\lambda}\in\Lambda^r} \bigoplus_{\psi=\phi\in\lceil\vec{\lambda}\rfloor} \mathbb{C}\chi_{\vec{\lambda}}^{\psi,\phi}. \quad \square$$

The Diagrammatic Basis

Steps to the Diagrammatic Representation.

1. Represent $\mathbb{C}\chi_\lambda$ diagrammatically.
2. Represent the injections $V_\psi, V_\phi \hookrightarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$ diagrammatically.
3. Combine the injections, the $G^{\times r}$-action, and the trace property.

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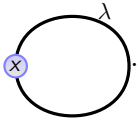
$$v^* \otimes w \mapsto \mathrm{tr}(x \mapsto v^*(x \cdot w)) = \sum_i v_i^*(x \cdot v_i).$$

The corresponding diagram is $\mathbb{C}\chi_\lambda =$ 

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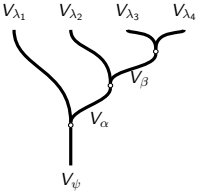
Each such injection corresponds to a term in the decomposition of this tensor product into irreducible elements.



In this diagram, each node represents an injection $V_\alpha \hookrightarrow V_\beta \otimes V_\gamma$, and the tree gives a well-defined way to perform this decomposition.

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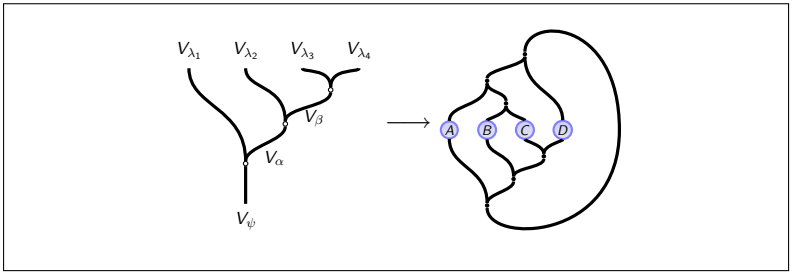

Central Functions on Arbitrary Surfaces

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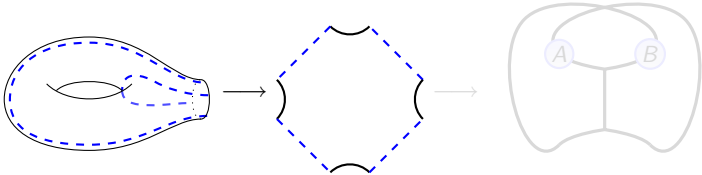

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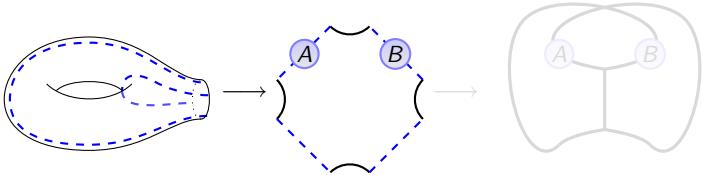

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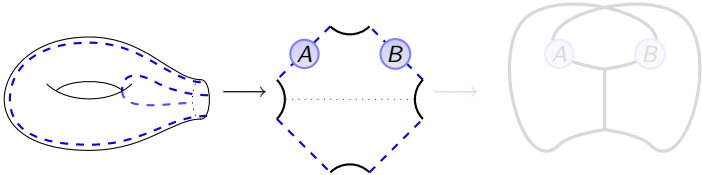

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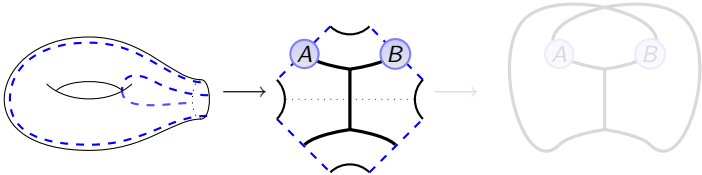

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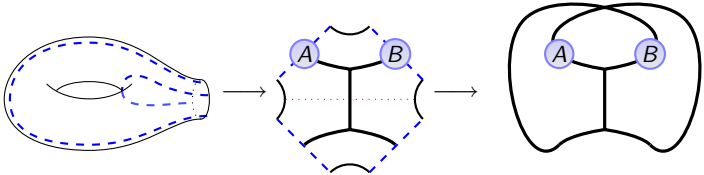
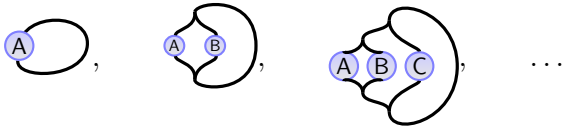


General Form of Central Functions

For this talk, we choose central functions of the form

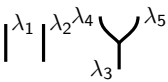


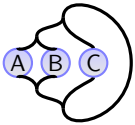
corresponding to π of rank $1, 2, 3, \ldots$

The set of diagrams of one type with admissible labelings comprises a basis for $\mathbb{C}[\mathfrak{X}]$.

Trace Diagrams and Spin Networks

A diagrammatic calculus for manipulating these functions already exists... spin networks!





- Strands correspond to representations of $\mathrm{SL}(2, \mathbb{C})$
- Vertices correspond to Clebsch-Gordan injections or projections
- Closed diagrams are functions $G \times G \times \cdots \times G \to \mathbb{C}$

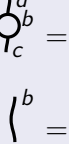
Spin Network Relations I

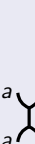
Define $\Theta(a,b,c) =$

and $\Delta(c) =$

.

Proposition (Bubble, Fusion Relations)



$= \dfrac{\Theta(a,b,c)}{\Delta(c)}$ 

$\delta_{c,d}$

$= \sum_{c \in \lceil a,b \rfloor} \dfrac{\Delta(c)}{\Theta(a,b,c)}$

where $\lceil a,b \rfloor = \{|a-b|, |a-b|+2, \ldots, a+b\}$ *is the set of values for which injections* $V_c \to V_a \otimes V_b$ *exist.*

Spin Network Relations II: Recoupling

Definition

The $6j$-*symbols* are the coefficients in the following change-of-basis equation:

$$= \sum_{f \in \lceil a,b \rfloor \cap \lceil c,d \rfloor} \begin{bmatrix} a & b & d \\ c & e & f \end{bmatrix}$$

.

Proposition (Gluing Identity)

$$= \sum_{\substack{a' \in \lceil 1,a \rfloor \\ c' \in \lceil 1,c \rfloor}} \mathfrak{F}_b \begin{pmatrix} a' & c' \\ a & c \end{pmatrix}$$

.

(This is an alternate normalization of the $6j$-symbol.)

The Simple Recurrence Formulas

Theorem

Let $\underline{a}$ denote $a-1$ and let γ be a loop along the trace diagram, and let $\mathsf{s}_{\gamma(a_i)}$ denote the trace diagram with labels a_i along the loop γ. Then

$$\mathsf{s}_{\gamma(a_i)} = \gamma \cdot \mathsf{s}_{\gamma(\underline{a}_i)} - \sum_{\substack{\{\underline{a}'_i = \underline{a}_i \pm 1\} \\ \text{some } \underline{a}'_i \neq a_i}} \left(\prod_{i=1}^{n} \pm \hat{\mathfrak{F}}_{b_i} \begin{pmatrix} \underline{a}'_{i-1} & \underline{a}'_i \\ \underline{a}_{i-1} & \underline{a}_i \end{pmatrix} \right) \mathsf{s}_{\gamma(\underline{a}'_i)}. \tag{1}$$

Proof.

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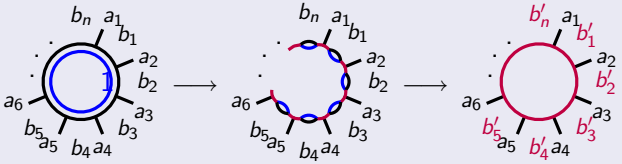


□

A Concrete Example

Example

$$= \hat{\mathfrak{F}}_b\left(\begin{smallmatrix}\bar{a} & \bar{a}\\ a & a\end{smallmatrix}\right)$$

$$- \hat{\mathfrak{F}}_b\left(\begin{smallmatrix}\underline{a} & \underline{a}\\ \underline{a} & \underline{a}\end{smallmatrix}\right)$$

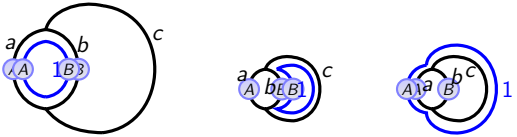
$$= $$

$$+ \frac{(a-\frac{b}{2})(a+\frac{b}{2}+1)}{a(a+1)}$$

$$. \qquad (2)$$

Rank Two Recurrences

There are three simple loops in the rank two case, corresponding to multiplication by $\mathrm{tr}(AB^{-1})$, $\mathrm{tr}(A)$, and $\mathrm{tr}(B)$:



Theorem (Rank Two Recursion)

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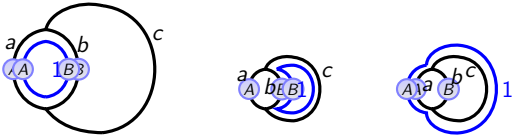

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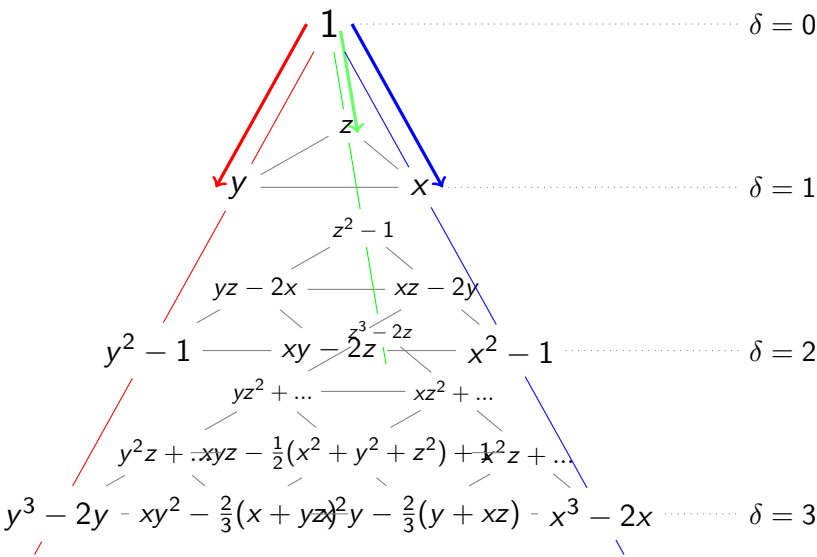
Rank Two: Table

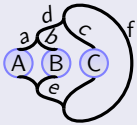


Rank Three Central Functions

Definition

The rank three central functions for $\mathbb{C}[\mathfrak{X}_3]$ are:

$$\chi^{a,b,c}_{d,e,f}(A,B,C) =$$



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where the triples $\{a,b,d\}$, $\{a,b,e\}$, $\{c,d,f\}$, and $\{c,e,f\}$ are all admissible.

There are six simple loops, but also a non-simple loop:

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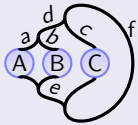

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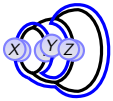
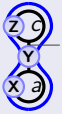
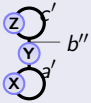


The Barbell

Proposition



$$= \sum_{\substack{a'=a\pm1\\ b'=b\pm1\\ c'=c\pm1\\ b''=b'\pm1}} \pm\hat{\mathfrak{F}}_a \begin{pmatrix} a' & b' \\ a & b \end{pmatrix} \hat{\mathfrak{F}}_c \begin{pmatrix} c' & b' \\ c & b \end{pmatrix} \hat{\mathfrak{F}}_{a'} \begin{pmatrix} a' & b'' \\ a & b' \end{pmatrix} \hat{\mathfrak{F}}_{c'} \begin{pmatrix} c' & b'' \\ c & b' \end{pmatrix}$$



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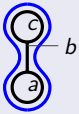
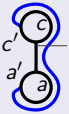


$\longrightarrow \sum$



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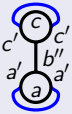

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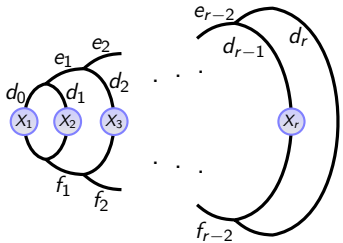
$\square$

The General Case

$\chi_{\vec{d},\vec{e},\vec{f}} \equiv$

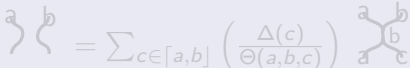


What else can you do with spin network relations?

Proposition (Rank One Product)

$$\chi_a \cdot \chi_b = \sum_{c \in \lceil a,b \rfloor} \chi_c.$$

Proof.

The relation  $= \sum_{c \in \lceil a,b \rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)} \right)$ implies

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Pull the matrix through the node and apply the bubble identity:

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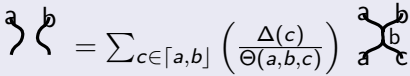
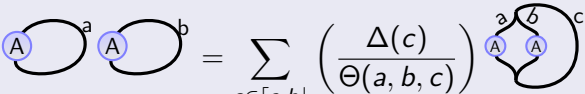
The relation  $= \sum_{c \in \lceil a,b \rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)} \right)$ implies



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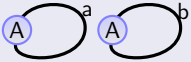

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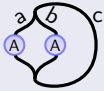

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$$\chi_c^{a,b}\chi_{c'}^{a',b'} = \sum_{j_1,j_2,k,l,m} C_{j_1,k,l,m}C_{j_2,k,l,m}\frac{\Theta(a,a',k)\Theta(b,b',l)\Theta(c,c',m)}{\Delta(k)\Delta(l)\Delta(m)}\chi_k^{l,m}$$

Idea of Proof.

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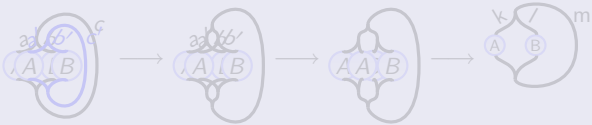

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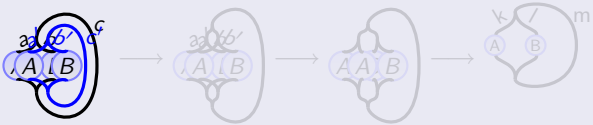

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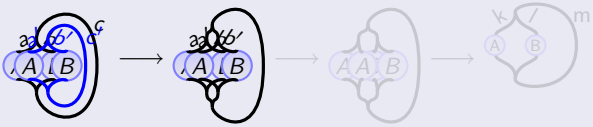

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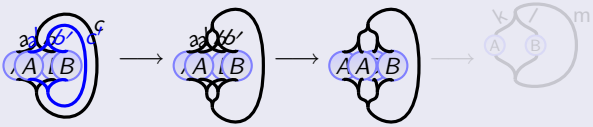

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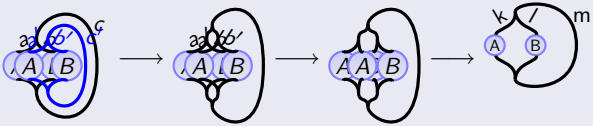


□

Questions??

References:

- Baez, *Spin networks in gauge theory*, **Adv. Math.** (1996)
- Sikora, *SL(n)-character varieties as spaces of graphs*, **Trans. AMS** (2001)
- Carter/Flath/Saito, *The Classical and Quantum 6j-Symbols*
- Lawton/Peterson, *Spin networks and SL(2,C)-character varieties*, **arXiv:math/0511271** (to appear, Handbook of Teichmüller Theory Vol II)
- Current work begun at KAIST Geometric Topology Fair, Summer 2007; preprint coming soon.

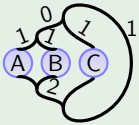
Generalizations

There are two obvious generalizations:

- Beyond rank two: work underway;
- Beyond $SL(2, \mathbb{C})$: requires new ideas since crossings cannot be removed very easily; will be much harder!

Rank Three: Sample Computation

Example

Compute $\chi^{1,1,1}_{0,2,1} =$ 

.

Rank Three: Partial Table

The interesting cases are when $a, b, c \neq 0$; otherwise, the functions reduce to rank two central functions. Also, either $d \neq 0$ or $e \neq 0$; otherwise, the diagram is disconnected.

Let $x = \mathrm{tr}(A)$, $y = \mathrm{tr}(B)$, $z = \mathrm{tr}(C)$, $X = \mathrm{tr}(B\bar{C})$, $Y = \mathrm{tr}(A\bar{C})$, and $Z = \mathrm{tr}(A\bar{B})$.

Case $a = b = c = 1$**:**

- $\chi^{1,1,1}_{0,2,1} = \frac{1}{2}zZ - X$
- $\chi^{1,1,1}_{2,0,1} = \frac{1}{2}zZ - [AC\bar{B}]$
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Case $a = b = c = 2$**:**

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Central Functions and Surface Structure

Question. How can trace diagrams take into account the structure of the *surface* as well as its fundamental group?

Partial Answer. Use the Poisson structure!

Definition

The Goldman bracket $\{f, g\}$ of two loops on a surface is the sum over all essential intersections of the following:

- This bracket satisfies the Jacobi and Leibniz identities, and so gives the ring a Poisson structure.
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Question. Can the theory of central functions be developed using Lie algebras rather than Lie groups?

Answer. Yes... I think! One interesting fact is that there is a nice diagram for transforming a matrix $X \in \mathrm{SL}(2, \mathbb{C})$ into a matrix $x \in sl(2, \mathbb{C})$, since the result necessarily has trace 0. This is the mapping

$X \mapsto X \frac{1}{2}\mathrm{tr}(X)I$. Such diagrams are the primary "building blocks" of one type of central function:

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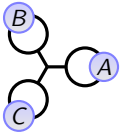

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