Trace Diagrams, Spin Networks, and Spaces of Graphs

KAIST Geometric Topology Fair

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United States Military Academy

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Motivation I

Every representation $\rho \in Hom(\pi, G)$ where $G = \mathrm{SL}(2, \mathbb{C})$ induces a map $\pi \to \mathbb{C}$ given by taking the trace of $\rho(x)$. These elements are conjugation-invariant and reside in both $\mathbb{C}[G^{\times r}]^G$ and the *coordinate ring of the character variety*.

$$\begin{tikzcd}
\pi \arrow[rr] \arrow[dr] & & \mathbb{C}\left[G^{\times r}\right]^G = \mathbb{C}[\chi] \\
x \arrow[rr, mapsto] \arrow[dr, mapsto] & ?? \arrow[ur] & \mathrm{tr}(\rho(x)) \\
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\end{tikzcd}$$

For this reason, the study of the structure of the character variety and its coordinate ring often boils down to examining trace relations [described in detail in Lawton's first talk]. This talk will describe a more geometric way to discuss trace relations.

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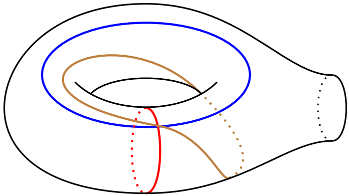

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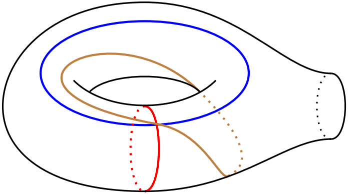

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Motivation II

We will develop an algebra which reflects both the structure arising from the fundamental group and that arising from its representation.

We call the requisite algebra the *Trace Diagram Algebra* T^2, whose elements are a special class of graphs marked by elements of $\mathrm{SL}(2, \mathbb{C})$.

$$\begin{tikzcd}
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\begin{tikzcd}
x \arrow[rr, mapsto] \arrow[dr, mapsto] & & \mathrm{tr}(\rho(x)) = \mathcal{G}(\mathrm{t}) \\
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The purpose of this talk is to describe $\mathcal{F}$ and $\mathcal{G}$, hence providing an alternate category for studying the coordinate ring of the character variety. This alternate category simplifies many calculations, and provides greater intuition for both trace polynomials and their connection with the character variety.

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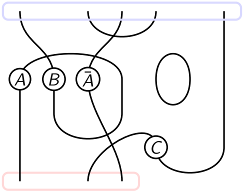
Outline

Definition of Trace Diagrams

Definition

A *2-trace diagram* $t \in T_2$ is a graph drawn in a box whose edges are marked by matrices in $M_{2\times 2}$. All vertices have degree one and occur at the bottom of the box (inputs), or at the top of the box (outputs). The diagrams are in **general position** relative to a certain "up" direction.



Note. $\bar{A} = A^{-1}$.

We use **general position** to mean the following:

- Each strand is an embedding;
- Crossings and matrix markings are disjoint from local extrema;
- Diagrams are equivalent if isotopic, provided the previous condition remains true and local extrema are neither added nor removed.

Diagrams with compatible inputs/outputs may be composed by placing one atop another. This corresponds to composition of functions.

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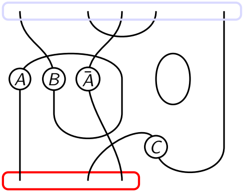

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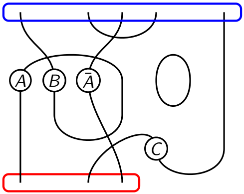

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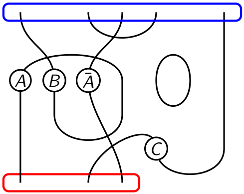

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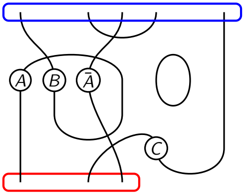

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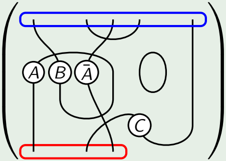
The Trace Diagram Functor

First we will construct, for $V = \mathbb{C}^2$ the standard representation of $\mathrm{SL}(2, \mathbb{C})$,

$$\mathrm{T}_2 \xrightarrow{\mathcal{G}} Fun(\underbrace{V \otimes \cdots \otimes V}_{i} \to \underbrace{V \otimes \cdots \otimes V}_{o}),$$

where i is the number inputs and o the number of outputs.

Example

$\mathcal{G}$  $: \quad V^{\otimes 3} \longrightarrow V^{\otimes 5}$

Component Decomposition of Trace Diagrams

The functor $\mathcal{G}(t)$ for a given trace diagram t will be defined by piecing together the action of $\mathcal{G}$ on smaller components.

Proposition

Every strand of a trace diagram may be uniquely decomposed into the components $|$, $\times$, $\underset{|}{\overset{|}{(A)}}$, $\cap$, and $\cup$.

Uniqueness comes from our restriction on "general position."

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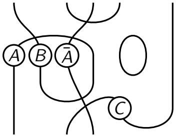

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Trace Diagram Component Maps

Definition

Given $v, w \in \mathbb{C}^2$, $A \in M_{2\times 2}$, and the standard basis $\{e_1, e_2\}$ of $\mathbb{C}^2$,

- Identity $\quad \mathcal{G}\left(\,|\,\right) : V \to V \quad$ takes $\quad v \mapsto v$
- Group Action $\quad \mathcal{G}\left(\textcircled{\scriptsize A}\right) : V \to V \quad$ takes $\quad v \mapsto Av$
- Permutations $\quad \mathcal{G}\left(\times\right) : V \otimes V \to V \otimes V \quad$ takes $\quad v \otimes w \mapsto w \otimes v$
- "Cap" $\quad \mathcal{G}\left(\cap\right) : V \otimes V \to \mathbb{C} \quad$ takes $\begin{cases} e_1 \otimes e_1 \mapsto 0 \\ e_1 \otimes e_2 \mapsto +1 \\ e_2 \otimes e_1 \mapsto -1 \\ e_2 \otimes e_2 \mapsto 0 \end{cases}$

  or $a \otimes b \mapsto \det[a \;\; b]$
- "Cup" $\quad \mathcal{G}\left(\cup\right) : \mathbb{C} \to V \otimes V \quad$ takes $\quad 1 \mapsto e_1 \otimes e_2 - e_2 \otimes e_1$

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Group Invariance

Theorem

The image of $\mathcal{G}$ lies in the set of multilinear functions $V^{\otimes i} \to V^{\otimes j}$ which are invariant with respect to simultaneous conjugation of all matrix elements by any $X \in \mathrm{SL}(2,\mathbb{C})$. In other words, for all $X \in \mathrm{SL}(2,\mathbb{C})$

$$X \cdot \mathcal{G}(t(A_1,\ldots,A_r)) \cdot \bar{X} = \mathcal{G}(t(XA_1\bar{X},\ldots,XA_r\bar{X})).$$

Here, $\cdot$ represents the action $X \cdot (v \otimes w) = Xv \otimes Xw$.

Remark. As a corollary, closed trace diagrams are *invariant under simultaneous conjugation in the matrix variables!*

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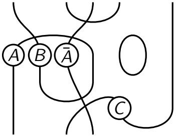
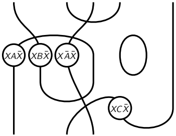
$X \cdot$ 

$\cdot \bar{X} =$ 

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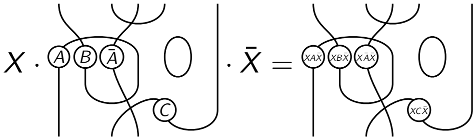

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Group Invariance: Proof

Proof.

It suffices to verify this property for the component maps. If $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$, then

$$X \cdot \mathcal{G}(\bigcup) \cdot \bar{X} = X \cdot \mathcal{G}(\bigcup) \qquad \text{takes}$$

$$\begin{aligned} 1 \mapsto X \cdot (e_1 \otimes e_2 - e_2 \otimes e_1) \\ = (x_{11}e_1 + x_{21}e_2) \otimes (x_{12}e_1 + x_{22}e_2) \\ - (x_{12}e_1 + x_{22}e_2) \otimes (x_{11}e_1 + x_{21}e_2) \\ = (x_{11}x_{22} - x_{12}x_{21})(e_1 \otimes e_2 - e_2 \otimes e_1) \\ = e_1 \otimes e_2 - e_2 \otimes e_1 = \mathcal{G}(\bigcup). \end{aligned}$$

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The proof for the cap $\bigcap$ is similar, while the calculations for the remaining component maps are trivial. $\square$

Trace Diagram Relations

Several diagrammatic relations arise from the component definitions, and it is easy to find two diagrams for which $\mathcal{G}(t_1) = \mathcal{G}(t_2)$. For example:

Proposition

$$\mathcal{G}\left(\bigcap\!\bigcup\right) = \mathcal{G}\left(-\diagup\right)$$

Proof.

Let $v = v_1 e_1 + v_2 e_2 \in \mathbb{C}^2$. Then

$$\begin{aligned}\mathcal{G}\left(\bigcap\!\bigcup\right)(v) &= \left(\bigcap \otimes \big|\right) \circ \left(\big| \otimes \bigcup\right)(v)\\ &= \left(\bigcap \otimes \big|\right)(v \otimes e_1 \otimes e_2 - v \otimes e_2 \otimes e_1)\\ &= (-v_2)e_2 - (v_1)e_1 = -v.\end{aligned}$$

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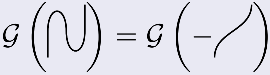

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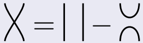
The Fundamental Binor Identity

Proposition (Fundamental Binor Identity)



Proof.

1.  represents the map $a \otimes b \mapsto b \otimes a$
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$$e_2 \otimes e_1 \overset{\cap}{\mapsto} -1 \overset{\cup}{\mapsto} -(e_1 \otimes e_2 - e_2 \otimes e_1) = e_2 \otimes e_1 - e_1 \otimes e_2 \qquad \checkmark$$

4. remaining basis elements work similarly $\square$

Remark. The binor identity provides a means of eliminating all crossings in an $\mathrm{SL}(2,\mathbb{C})$ trace diagram!

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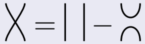

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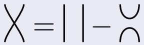

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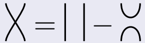

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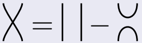

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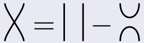

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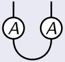
Matrices at Critical Points

Proposition (Critical Points)

Matrices pass through critical points via



$= \det(A)$

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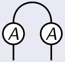

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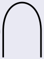
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Proof.

The definition $\det A = a_{11}a_{22} - a_{12}a_{21}$ becomes the diagram

$\det(A) =$

$\square$

Corollary

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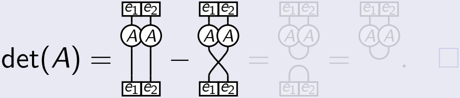

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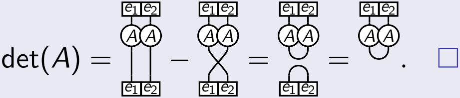

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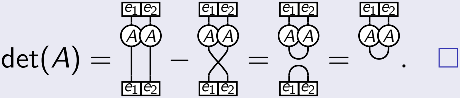

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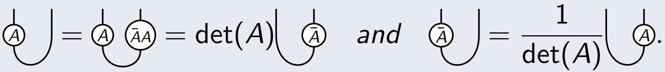


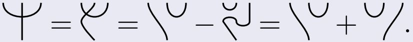
Crossings at Local Extrema

Proposition

Show that the diagram  *may be well-defined.*

Proof.

Move the crossing away from the extremum:



Using  $=$  would have provided the same answer. $\square$

Remark: This makes the condition that crossings are disjoint from local extrema unnecessary.

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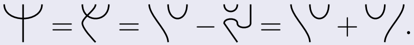
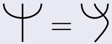


Using  would have provided the same answer. □

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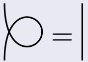
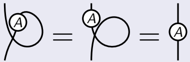
Looping Relation

Proposition (Looping Relation)

Arcs may be wrapped around as follows:



and consequently



Proof.

Apply the algebraic definition or the binor identity in each case. □

Closed Diagrams: Trace Polynomials

Closed diagrams with matrices can be thought of as

- Functions $G^{\times r} \to \mathbb{C}$ invariant under simultaneous conjugation;
- Trace polynomials.

Proposition (Trace)

$$= \mathrm{tr}(A) = a_{11} + a_{22}.$$

Corollary

$= \mathrm{tr}(I) = 2$ *and*

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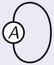
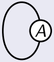
 $=$  $= \mathrm{tr}(A) = a_{11} + a_{22}.$

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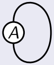
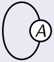
 $=$  $= \mathrm{tr}(A) = a_{11} + a_{22}.$

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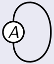 $= \bigcap \circ A \otimes I \circ \bigcup : 1 \mapsto Ae_1 \otimes e_2 - Ae_2 \otimes e_1$

$$= (a_{11}e_1 + a_{21}e_2) \otimes e_2 - (a_{12}e_1 + a_{22}e_2) \otimes e_1$$

$$\mapsto a_{11} + a_{22}.$$

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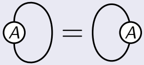

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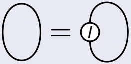

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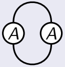

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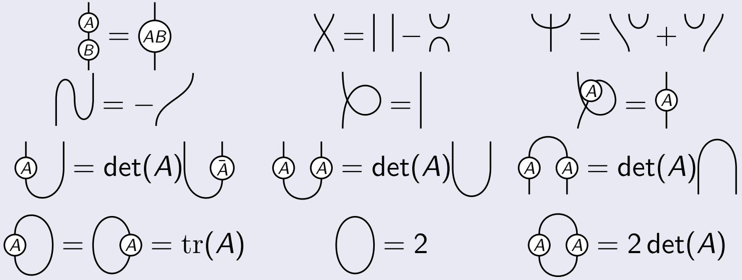
Summary of Diagram Rules

Diagram Rule (Summary)



Example I

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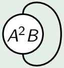
Use the binor identity to reduce $\mathrm{tr}(A^2B) = [A^2B] =$ .

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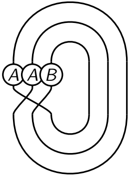

$=$

This corresponds to

$$\begin{aligned}[A^2B] &= [A]^2[B] - 2\det(A)[B] - \det(B)[A][A\bar{B}] + \det(A)[B] \\ &= [A]^2[B] - \det(A)[B] - \det(B)[A][A\bar{B}]. \quad \square\end{aligned}$$

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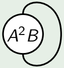
Use the binor identity to reduce $\mathrm{tr}(A^2B) = [A^2B] =$ .

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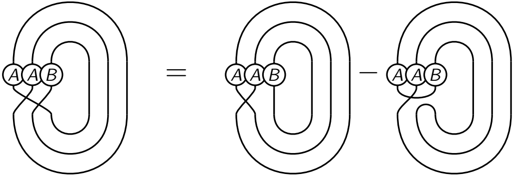

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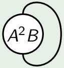
Use the binor identity to reduce $\mathrm{tr}(A^2B) = [A^2B] =$ .

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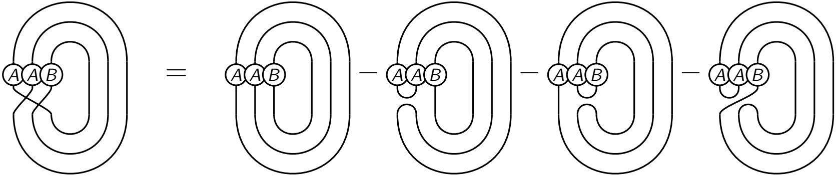

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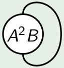
Use the binor identity to reduce $\mathrm{tr}(A^2B) = [A^2B] =$ .

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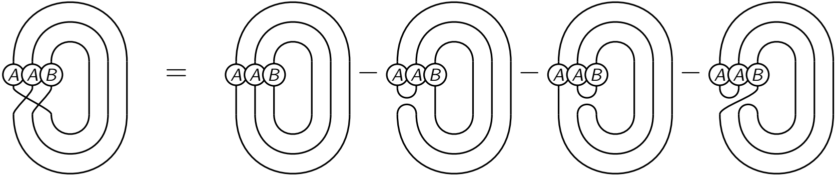

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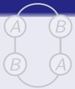
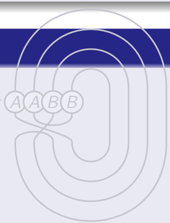
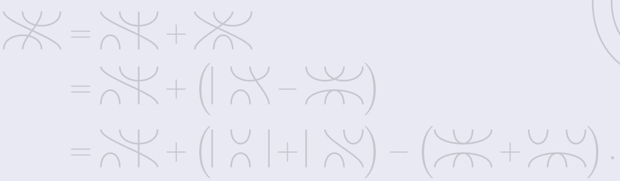
Example II: The Commutator Relation

Proposition

$$\mathrm{tr}(AB\bar{A}\bar{B}) = [AB\bar{A}\bar{B}] = [A]^2 + [B]^2 + [AB]^2 - [A][B][AB] - 2.$$

Proof.

Represent the trace diagrammatically as  $= -$  .



Re-insert matrices and keep track of signs:

$$[AB\bar{A}\bar{B}] = [B]^2 - [A][B][A\bar{B}] + [A]^2 + [A\bar{B}]^2 - 2.$$

Use $[A\bar{B}] = [A][B] - [AB]$ to obtain the desired relation. □

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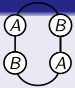
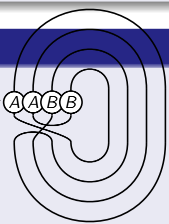
Represent the trace diagrammatically as  $= -$ .

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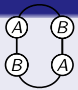
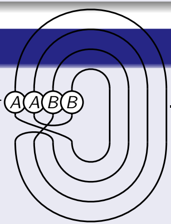
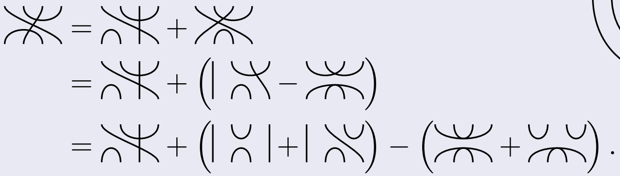
Represent the trace diagrammatically as  $= -$  .



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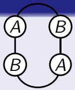
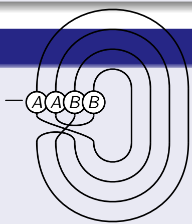
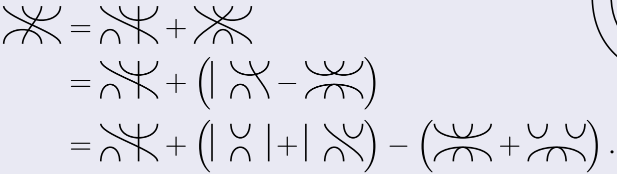
Represent the trace diagrammatically as  $= -$  .



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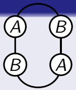
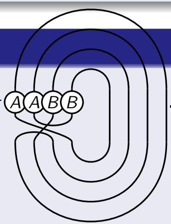
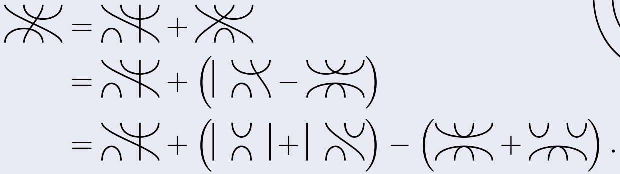
Represent the trace diagrammatically as  $= -$ .



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Example III: The 2×2 Characteristic Equation

The characteristic equation arises by replacing the eigenvalues in the *characteristic polynomial* $\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$ with the matrix.

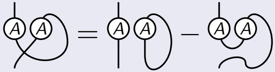
Proposition

The binor identity implies the characteristic equation

$$A^2 - \mathrm{tr}(A)A + \det(A)I = 0.$$

Proof.

By the binor identity,



This last expression is $A^2 = A * \mathrm{tr}(A) - \det(A)I$, the characteristic equation. $\square$

The Category Morphism

The next task is to construct $\mathcal{F}$ in the figure below.

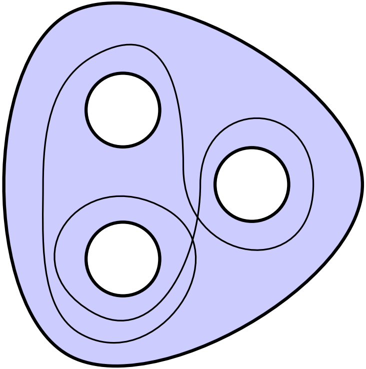
$$
\begin{tikzcd}
\pi \arrow[rr] \arrow[dr, "\mathcal{F}"] & & \mathbb{C}\left[G^{\times r}\right]^G \\
 & \mathrm{T}^2 \arrow[ur, "\mathcal{G}"] & \\
x \arrow[rr, mapsto] \arrow[dr, mapsto] & & \mathrm{tr}(\rho(x)) = \mathcal{G}(\mathrm{t}) \\
 & \mathrm{t} = \mathcal{F}(x) \arrow[ur, mapsto] &
\end{tikzcd}
$$

We will use this construction to examine the structure of the coordinate ring $\mathbb{C}[\chi]$.

Surface Group Representations

We now define the function $\mathcal{F} : \pi_1 \to \mathrm{T}_2$ which assigns a trace diagram to each element of the fundamental group.

- Assign 'surface cuts' to elements of the fundamental group.
- Mark loops at the cuts using the representation $\pi_1 \to G$.
- Ensure drawing is compatible with an "up" direction.



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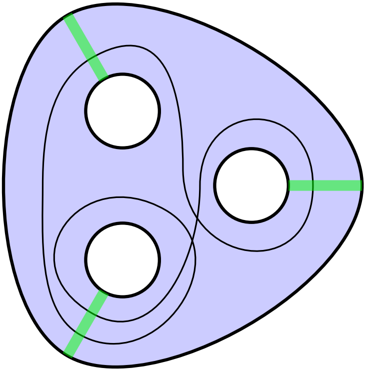

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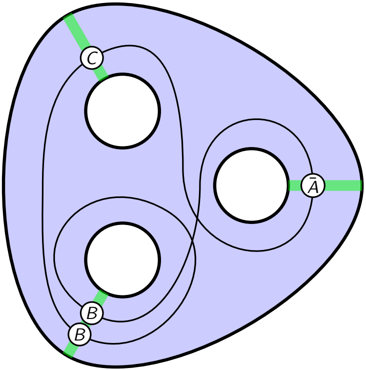

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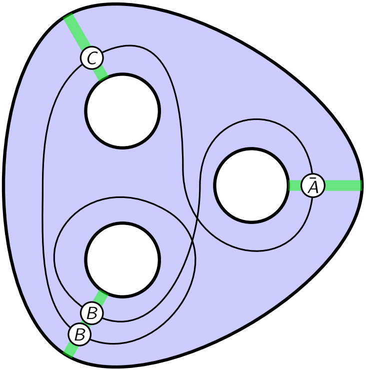

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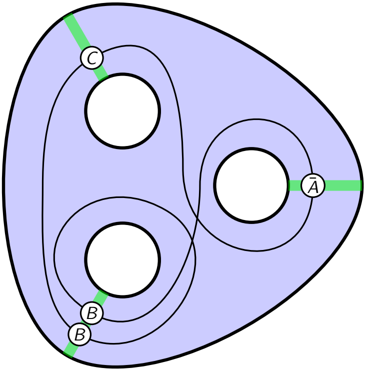

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The Character Variety

The ring of trace polynomials may be used to construct the following:

- The *G-character variety* $\mathfrak{X}$ is the algebraic variety whose coordinate ring is the trace ring generated by representations.

In other words, the space of trace diagrams on a surface can be thought of as precisely $\mathbb{C}[\mathfrak{X}]$.

Rank Two

Proposition

For surfaces with free group of rank two, the coordinate ring $\mathbb{C}[\chi]$ is a polynomial ring in three indeterminates.

Proof.

Given the binor identity , all trace loops on the three-holed sphere can be reduced to three basic loop types.

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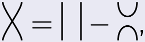

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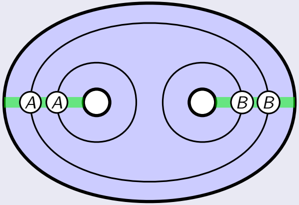

□

Beyond Crossing Removal

What happens when we remove all crossings in higher rank cases??

Problem. There are an infinite number of diagams without crossings!

Definition

A 2-trace diagram in a surface is *simple* if it has no self-crossings and passes through each cut set (or contains each matrix variable) at most once.

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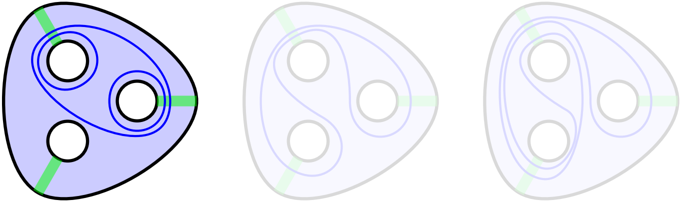

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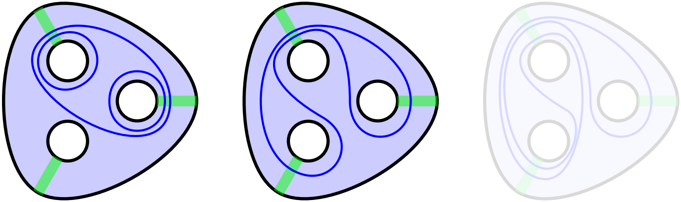

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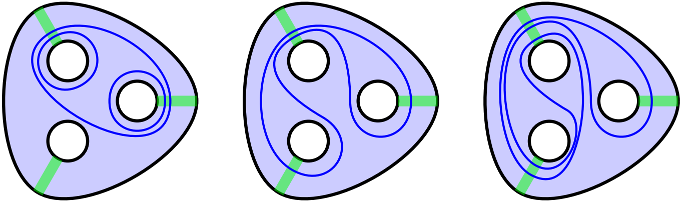

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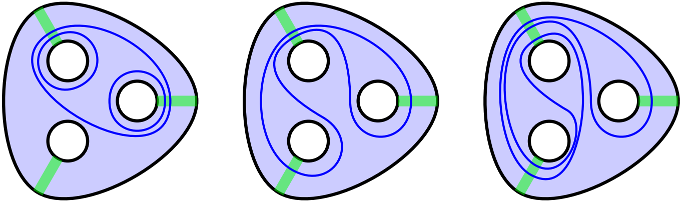

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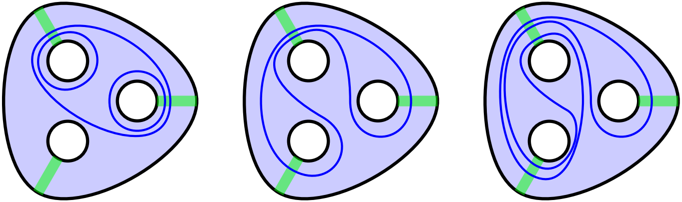

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Trace diagram relations can reduce any surface loop to simple diagrams.

Proof.

Use the binor identity to remove crossings. For duplicates:

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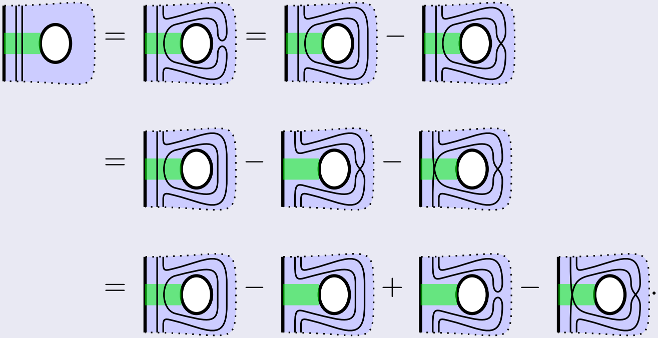

□

Reduction of Diagrams II

Remark. Algebraically, the ability to reducec is simply the statement that $\mathrm{SL}(2,\mathbb{C})$ trace relations can be used to reduce the polynomials for $\mathrm{tr}(A\cdots A\cdots)$ and $\mathrm{tr}(A\cdots \bar{A}\cdots)$.

4-Element Trace Relations

Proposition

$$\begin{aligned}2[ABCD] = {} & [A][B][C][D] + [AB][CD] + [BC][AD] - [AC][BD] \\ & - [A][B][CD] - [B][C][AD] - [C][D][AB] - [A][D][BC] \\ & + [A][BCD] + [B][CDA] + [C][DAC] + [D][ABC].\end{aligned}$$

Proof.

Reduce the crossings in the following diagram:

Now take the trace of all elements. The figure at left is $[AC][BD]$. The first five terms on the right contribute to the $2[ABCD]$ term. The remaining 11 terms are the rest of the relation. □

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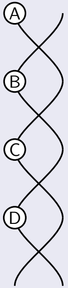

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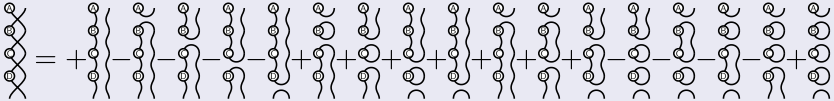

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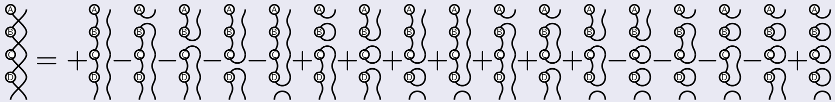

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We have now proven:

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The set of simple trace diagrams with no more than three elements generates the space of all closed trace diagrams on a surface.

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$$[ABC] + [CBA] = [A][BC] + [B][AC] + [C][AB] - [A][B][C]$$

Proof.

The anti-symmetrizer sends any $a \otimes b \otimes c \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ to zero. Given that , this implies the relations

This is precisely the identity above. $\square$

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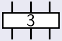
The anti-symmetrizer  sends any $a \otimes b \otimes c \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ to zero. Given that , this implies the relations

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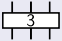
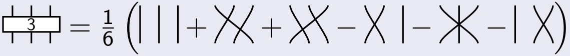
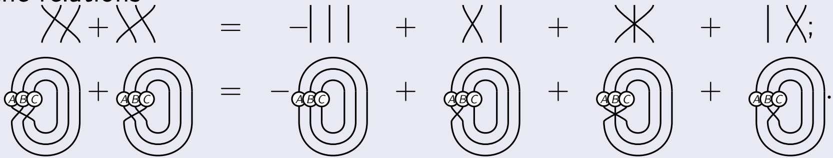
The anti-symmetrizer  sends any $a \otimes b \otimes c \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ to zero. Given that  , this implies the relations



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The crux of the argument comes in reducing the crossings of the following diagram, which when closed gives the product $[ABC][CBA]$:

The result will be a diagam with sixteen terms including loops for elements such as $[A\bar{B}]$ or $[AB]$. Applying the relation $[A\bar{B}] = [A][B] - [AB]$ reduces the result to the above form. □

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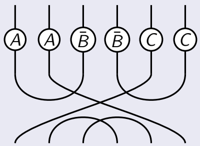

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Some Invariant Theory Results

Theorem

A minimal generating set of $\mathbb{C}[\chi]$ *consists of* $\{\mathrm{tr}(X_i)\}$, $\{\mathrm{tr}(X_iX_j)\}$ *for* $i<j$, *and* $\{\mathrm{tr}(X_iX_jX_k)\}$ *for* $i<j<k$.

Theorem

A maximal independent set of generators consists of $\{\mathrm{tr}(X_i)\}$ *and* $\{\mathrm{tr}(X_iX_j)\}$ *for* $j=i+1$ *or* $j=i+2$.

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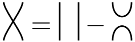
Why the Diagrammatic Approach?

Diagrams are good for:

- exhibiting mathematical structure
  - duality corresponds to turning diagrams upside-down
  - relations are often simply expressed: 
- connecting algebra with geometry
  - when placed on surfaces, trace diagrams describe the moduli space of representations of a surface group
- discovering similarities among mathematical structures
  - is both a 2×2 trace identity and the defining relation of the Poisson bracket on the coordinate ring of the character variety
- computational algorithms
  - relations used to generate recurrence equations;
  - illustrative method for generating trace identities.

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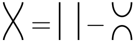
  - relations are often simply expressed: 
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  - when placed on surfaces, trace diagrams describe the moduli space of representations of a surface group
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  - is both a 2×2 trace identity and the defining relation of the Poisson bracket on the coordinate ring of the character variety
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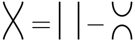
  - relations are often simply expressed: 
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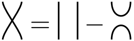
  -  is both a 2×2 trace identity and the defining relation of the Poisson bracket on the coordinate ring of the character variety
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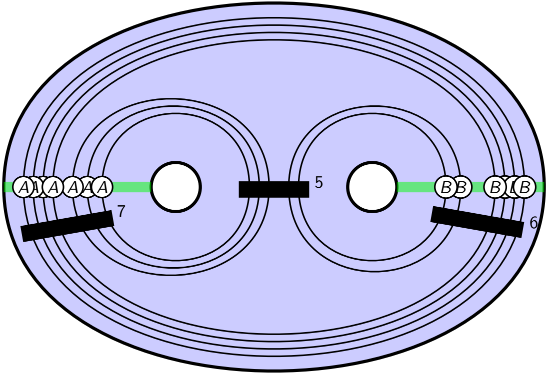
Basis for the Coordinate Ring I

Diagrams can be used to construct a basis for the trace ring.



Expand symmetrizers and remove crossings to obtain a trace polynomial

$$\chi_5^{7,6}(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(A\bar{B}))$$

Theorem

The polynomials $\chi_c^{a,b}$ comprise a basis for the coordinate ring of the $\mathrm{SL}(2,\mathbb{C})$-character variety of the three-holed sphere.

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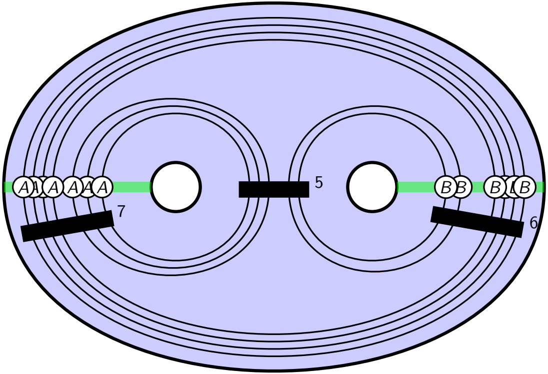

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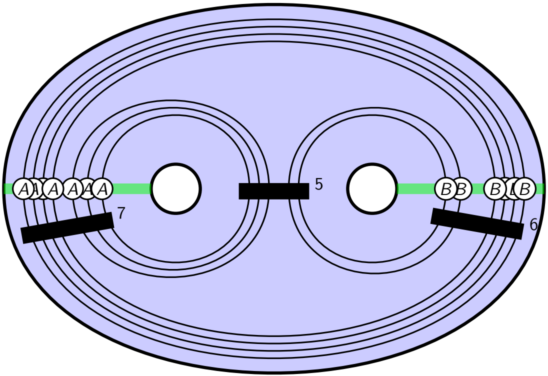

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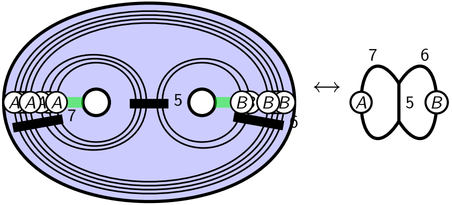
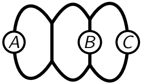
Basis for the Coordinate Ring II

Shorthand: 

Remarks:

- Edges are labeled by *representations*.
- The basis exhibits considerable symmetry.
- The basis depends only on fundamental group of the surface.
- To generalize for other surfaces, add more loops: 

Acknowledgments/References

- Bill Goldman
- Sean Lawton
- Charles Frohman, Doug Bullock, Joanna Kania-Bartoszynska.
- Carter/Flath/Saito, *The Classical and Quantum 6j-Symbols*