# Diagrammatic Central Functions

## KAIST Geometric Topology Fair

Elisha Peterson

United States Military Academy

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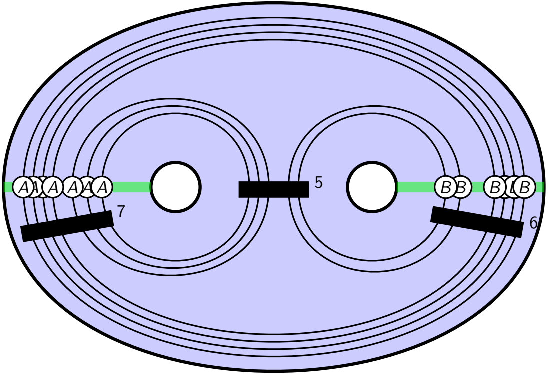
# Outline

# Basis for the Coordinate Ring I

Diagrams can be used to construct a basis for the trace ring.




**Theorem**

*The polynomials $\chi_c^{a,b}$ comprise a basis for the coordinate ring of the $\mathrm{SL}(2,\mathbb{C})$-character variety of the three-holed sphere.*

- Proof uses the unitary trick and the Peter-Weyl Theorem.

Expand symmetrizers and remove crossings to obtain a trace polynomial

$$\chi_5^{7,6}(\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(A\bar{B}))$$

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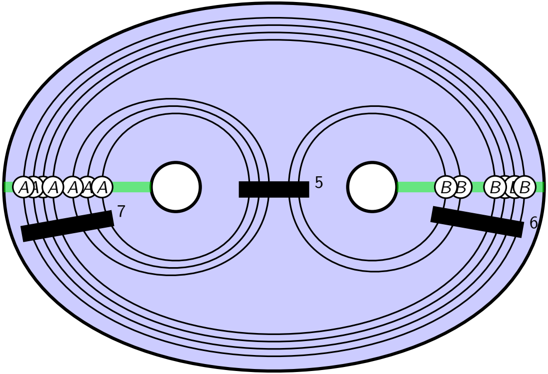


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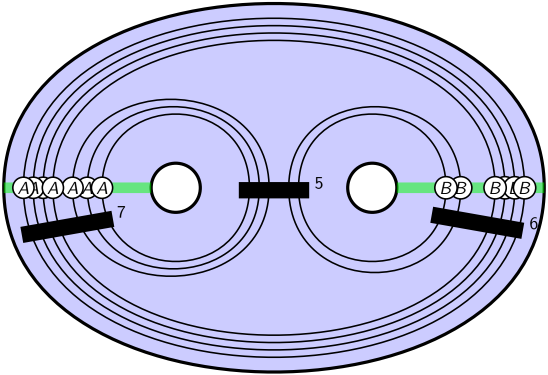


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# The Peter-Weyl Theorem

The Peter-Weyl Theorem provides a means to describe a basis of functions for a coordinate ring.

## Theorem (Corollary of Peter-Weyl)

*The coordinate ring $\mathbb{C}[G]$ for a reductive linear algebraic group $G$ decomposes:*

$$\bigoplus_{\lambda \in \Lambda} V_\lambda^* \otimes V_\lambda \cong \mathbb{C}[G],$$

*where $\Lambda$ is the set of irreducible representations (of the maximal compact subgroup $U \subset G$), and the isomorphism is given by*

$$v^* \otimes w \mapsto (x \mapsto v^*(x \cdot w)) .$$

# Central Functions of the Character Variety

**Theorem (Central Function Decomposition)**

*The coordinate ring of the character variety may be decomposed*

$$\mathbb{C}[\mathfrak{X}_r] \cong \bigoplus_{\vec{\lambda}\in\Lambda^r} \bigoplus_{\psi=\phi\in\lceil\vec{\lambda}\rfloor} \mathbb{C}\chi_{\vec{\lambda}}^{\psi,\phi},$$

*where* $\psi = \phi \in \lceil\vec{\lambda}\rfloor$ *indicates that* $V_\psi, V_\phi \hookrightarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$, *but may possibly be different injections.*

**Definition (Central Functions)**

The *Central Functions* of the *G*-character variety $\chi_r$ are the functions $\chi_{\vec{\lambda}}^{\psi,\phi}$ in the above decomposition.

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# Proof of the Central Function Decomposition

## Proof.

When a surface $\Sigma$ has fundamental group *free* of rank $r$, the isomorphism $\mathbb{C}[Hom(\pi, G)] \cong \mathbb{C}[G^{\times r}] \cong \mathbb{C}[G]^{\otimes r}$ and the previous result give:

$$\mathbb{C}[\mathfrak{X}] \cong \left(\mathbb{C}[G]^{\otimes r}\right)^G \cong \left(\bigotimes_r \bigoplus_{\lambda\in\Lambda} V_\lambda^* \otimes V_\lambda\right)^G$$

$$\cong \bigoplus_{(\lambda_1,\dots,\lambda_r)\in\Lambda^r} \left(\left(V_{\lambda_1}^* \otimes \cdots \otimes V_{\lambda_r}^*\right) \otimes \left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}\right)\right)^G.$$

Schur's Lemma and $G$-invariance permit a reduction to the desired form:

$$\mathbb{C}[\mathfrak{X}_r] \cong \bigoplus_{\vec{\lambda}\in\Lambda^r} \bigoplus_{\psi=\phi\in\lceil\vec{\lambda}\rfloor} \mathbb{C}\chi_{\vec{\lambda}}^{\psi,\phi}. \qquad \square$$

# The Diagrammatic Basis

**Steps to the Diagrammatic Representation.**

1. Represent $\mathbb{C}\chi_\lambda$ diagrammatically.
2. Represent the injections $V_\psi, V_\phi \hookrightarrow V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}$ diagrammatically.
3. Combine the injections, the $G^{\times r}$-action, and the trace property.

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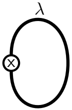
The isomorphism $V_\lambda^* \otimes V_\lambda \cong \mathbb{C}\chi_\lambda$ is defined for a basis $\{v_i\}$ of $V_\lambda$ by

$$v^* \otimes w \mapsto \mathrm{tr}(x \mapsto v^*(x \cdot w)) = \sum_i v_i^*(x \cdot v_i).$$

The corresponding diagram is $\mathbb{C}\chi_\lambda =$ 

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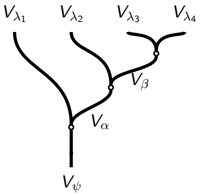
Each such injection corresponds to a term in the decomposition of this tensor product into irreducible elements.




In this diagram, each node represents an injection $V_\alpha \hookrightarrow V_\beta \otimes V_\gamma$, and the tree gives a well-defined way to perform this decomposition.

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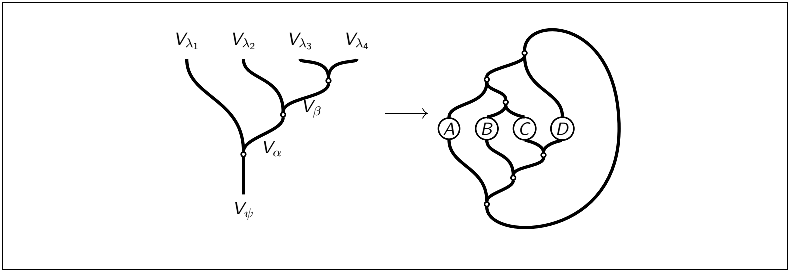
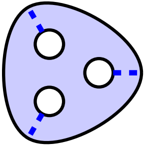


# Cut Triangulations



A *cut set* as defined in the previous talk permits “opening” up a surface onto the plane.



**Definition**

A *cut triangulation* is an extension of a cut set which divides the surface into a set of triangles (with neighborhoods of vertices removed).

Cut triangulations provide canonical decompositions of

$$\left(\left(V^*_{\lambda_1} \otimes \cdots \otimes V^*_{\lambda_r}\right) \otimes \left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_r}\right)\right)^G .$$

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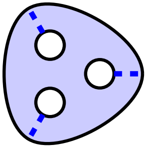

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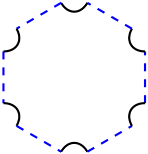

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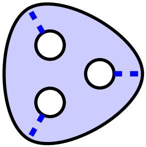

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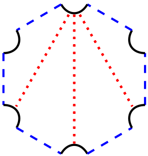

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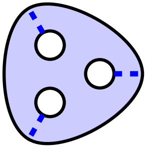

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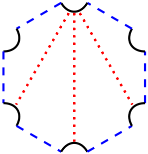

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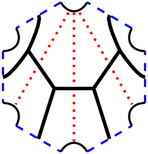
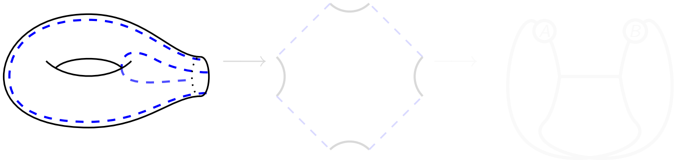


# The Diagrammatic Basis Theorem

## Theorem

*Let $\Sigma$ be a compact surface with boundary. Given a cut triangulation extending a specified cut set, every $G$-admissible labelling of its dual 1-skeleton induces a trace diagram which is identified with a $G$-invariant function $Hom(\pi, G) \to \mathbb{C}$. Moreover, for every cut triangulation, the set of such diagrams is a basis for $\mathbb{C}[\mathfrak{X}]$.*

# Example: the 1-Holed Torus

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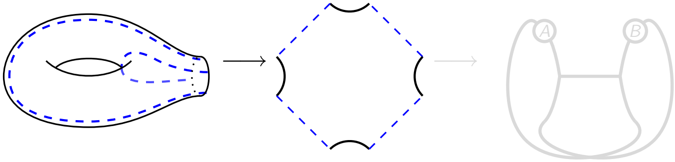


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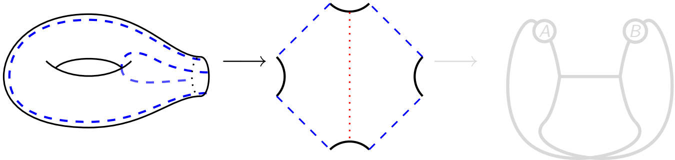


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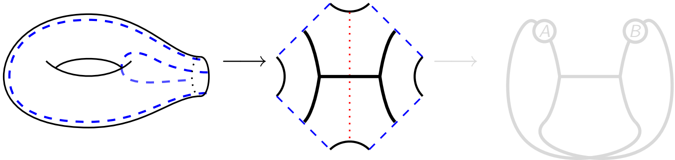


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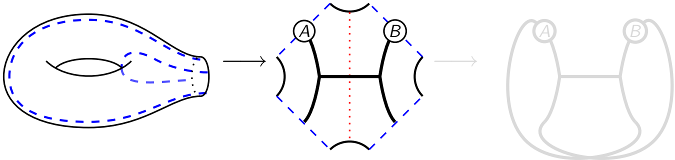


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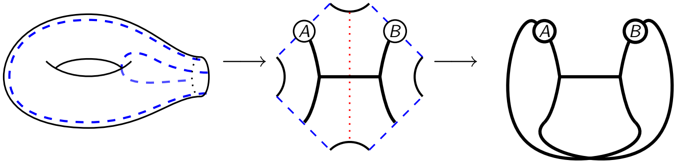


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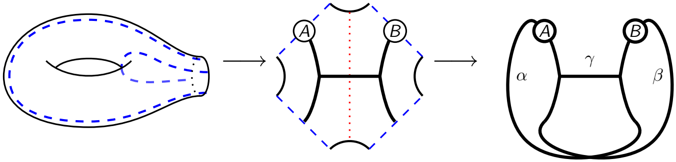


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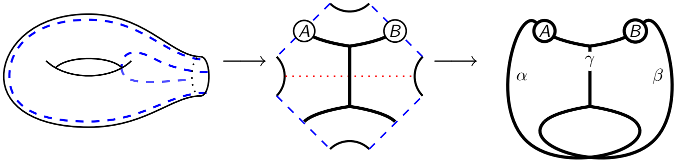


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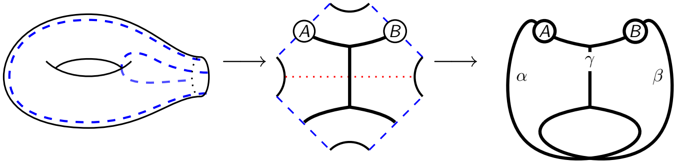


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# Representation Theory

Trick to working with trace diagrams marked by representations:

1. Use injections $V_\lambda \hookrightarrow V \otimes \cdots \otimes V$ to access a "copy" of the representation lying inside tensor algebra (*Young Projectors*);
2. Use the $n$-Trace Diagram Calculus to manipulate the diagrams.

For many Lie groups, all irreducible representations can be understood in this way:

- SL$(2,\mathbb{C})$: 

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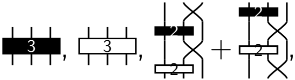
- SL$(3,\mathbb{C})$: 

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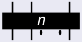
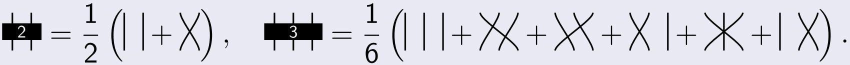
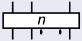
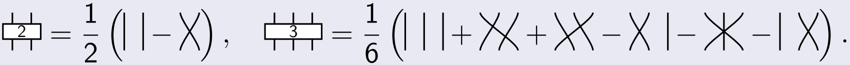
# Symmetrizers and Anti-Symmetrizers

## Definition

The *symmetrizer* map  : $V^{\otimes n} \to V^{\otimes n}$ is the normalized sum of all permutations. For example:



The *anti-symmetrizer* map  : $V^{\otimes n} \to V^{\otimes n}$ is the normalized sum of even permutations minus odd permutations. For example:



The symmetrizer  can be thought of as a map $Sym_n(V) \hookrightarrow V^{\otimes n}$, hence picks out a copy of the irreducible representation $V_n$ inside the tensor product.

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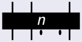
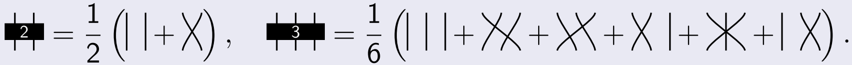
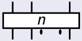
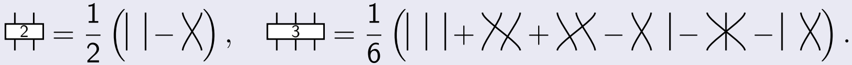
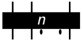
The *symmetrizer* map  : $V^{\otimes n} \to V^{\otimes n}$ is the normalized sum of all permutations. For example:



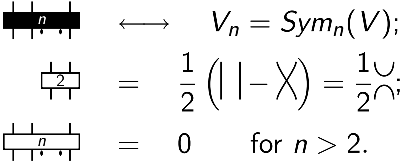
The *anti-symmetrizer* map  : $V^{\otimes n} \to V^{\otimes n}$ is the normalized sum of even permutations minus odd permutations. For example:



The symmetrizer  can be thought of as a map $Sym_n(V) \hookrightarrow V^{\otimes n}$, hence picks out a copy of the irreducible representation $V_n$ inside the tensor product.

# $SL(2,\mathbb{C})$ Representation Diagrams

When $G = \mathrm{SL}(2,\mathbb{C})$, the following are true:

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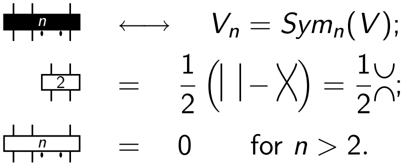


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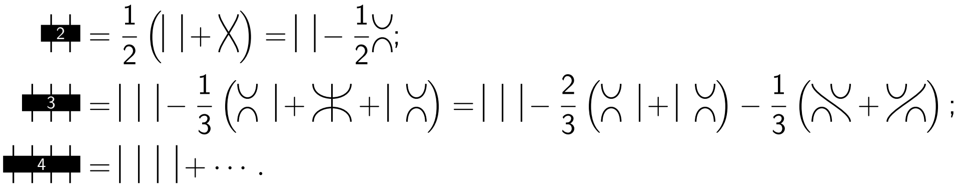


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# Rank One: Sample Computation

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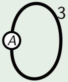
Compute $\chi_3(A) =$  .

**Solution.** Expand the symmetrizer as follows:



Apply the matrix and close off the terms to get:

$$\chi_3(A) = [A]^3 - \frac{1}{3}(3[A][A\bar{A}]) = [A]^3 - \frac{1}{3}(6[A]) = [A]^3 - 2[A]. \qquad \square$$

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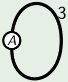
Compute $\chi_3(A) =$ .

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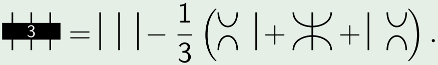

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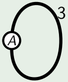
Compute $\chi_3(A) =$  .

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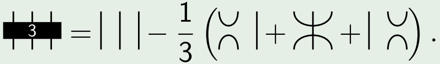
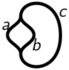


Apply the matrix and close off the terms to get:

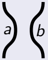
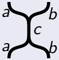
$$\chi_3(A) = [A]^3 - \frac{1}{3}(3[A][A\bar{A}]) = [A]^3 - \frac{1}{3}(6[A]) = [A]^3 - 2[A]. \quad \square$$

# Trivalent Trace Properties I

Define $\Theta(a,b,c) =$  and $\Delta(c) =$ . Then:

## Proposition (Bubble, Fusion Relations)

 $= \left( \dfrac{\Theta(a,b,c)}{\Delta(c)} \Big|^{c} \right) \delta_{cd};$

 $= \sum_{c \in \lceil a,b \rfloor} \left( \dfrac{\Delta(c)}{\Theta(a,b,c)} \right)$ ;

## Proposition (Recoupling)

 $= C$ 

*for some coefficient $C$ depending on $a,\dots,f$ called a 6j-Symbol.*

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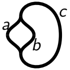
Define $\Theta(a,b,c) =$  and $\Delta(c) =$ . Then:

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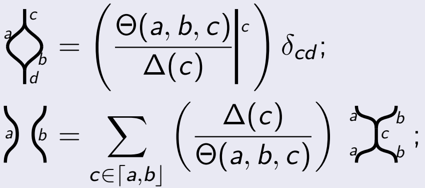


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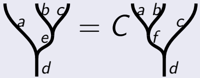


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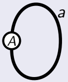
# Rank One

## Definition

The rank one central functions (corresponding to an annulus) are

$$\chi_a(A) =$$  .

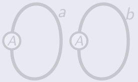
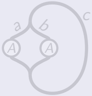
# Rank One: Ring Structure

## Proposition (Rank One Product Formula)

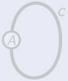
$$\chi_a \cdot \chi_b = \sum_{c \in \lceil a,b \rfloor} \chi_c.$$

## Proof.

The relation  $= \sum_{c\in\lceil a,b\rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)}\right)$  implies

 $= \sum_{c\in\lceil a,b\rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)}\right)$ .

Pull the matrix through the node and apply the bubble identity:

$\chi_a\chi_b = \sum_{c\in\lceil a,b\rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)}\right)\left(\frac{\Theta(a,b,c)}{\Delta(c)}\right)$ . □

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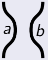
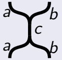
The relation 

$= \sum_{c\in\lceil a,b\rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)}\right)$ 

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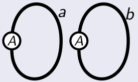


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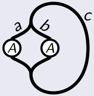


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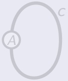


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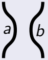
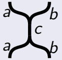
The relation 

$= \sum_{c\in\lceil a,b\rfloor} \left(\frac{\Delta(c)}{\Theta(a,b,c)}\right)$ 

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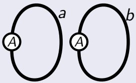


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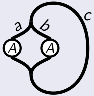

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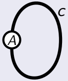

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# Rank One: Table

As a special case of the previous result,

$$x \cdot \chi_{a-1} = \chi_1 \cdot \chi_{a-1} = \chi_{a-2} + \chi_a.$$

Hence, the central functions satisfy the *Chebyshev* or *Fibonacci recurrence* $\chi_a = x \cdot \chi_{a-1} - \chi_{a-2}$ and are easily computed:

$$\begin{aligned}
\chi_0(A) &= 1 \\
\chi_1(A) &= x \\
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# Rank One: Other Properties

Some other results for rank one central functions:

- $x^n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{n}{r} - \binom{n}{r-1} \right) \chi_{n-2r}$, where $\binom{n}{r} = 0$ for $r \leq 0$;
- $\chi_n = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r \binom{n-r}{r} x^{n-2r}$;
- If $\lambda$ is an eigenvalue of $A$, then $\chi_n = [n+1]_\lambda$, the *quantized integer* with $q = \lambda$;
- If $\mathrm{tr}(A) = i = \sqrt{-1}$, then $\chi_n = i^n F_n$, where $F_n$ is the *nth Fibonacci number*.

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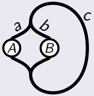
# Rank Two

## Definition

The rank two central functions for $\mathbb{C}[\mathfrak{X}_2]$ are: $\chi_c^{a,b}(A,B) =$ ,



where $\{a,b,c\}$ is any *admissible triple*.

**Alternate Parameters:** Define $\chi_{\alpha,\beta,\gamma} \equiv \chi_c^{a,b}$, where $\alpha \equiv \frac{1}{2}(-a+b+c)$, $\beta \equiv \frac{1}{2}(a-b+c)$, and $\gamma \equiv \frac{1}{2}(a+b-c)$. These are the number of each type of loop occurring when the symmetrizers are expanded. The admissibility condition is then $\alpha,\beta,\gamma \geq 0$.





We typically write $\chi_c^{a,b}$ in terms of $x = \mathrm{tr}(A), y = \mathrm{tr}(B), z = \mathrm{tr}(A\bar{B})$.

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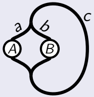
The rank two central functions for $\mathbb{C}[\mathfrak{X}_2]$ are: $\chi_c^{a,b}(A, B) =$ ,



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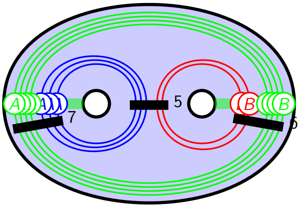



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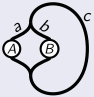
The rank two central functions for $\mathbb{C}[\mathfrak{X}_2]$ are: $\chi_c^{a,b}(A, B) =$ ,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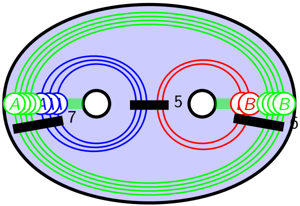


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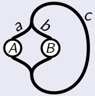
The rank two central functions for $\mathbb{C}[\mathfrak{X}_2]$ are: $\chi_c^{a,b}(A,B) =$ 

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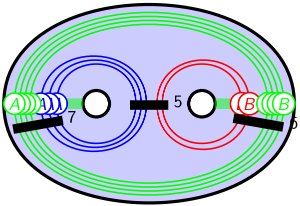


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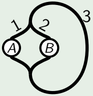
# Rank Two: Sample Computation

## Example

Compute $\chi_3^{1,2}(A,B) =$ .



**Solution.** Expand the symmetrizer as follows:





Apply the matrix and close off the terms to get:

$$\begin{aligned}\chi_3^{1,2}(A,B) &= [A][B]^2 - \frac{1}{3}([A\bar{B}][B] + [A\bar{B}][B] + 2[A]) \\ &= [A][B]^2 - \frac{2}{3}([A\bar{B}][B] + [A]). \quad \square\end{aligned}$$

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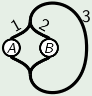
Compute $\chi_3^{1,2}(A,B) =$ .



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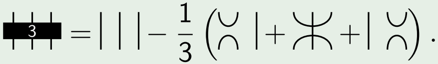



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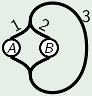
Compute $\chi_3^{1,2}(A, B) =$ .



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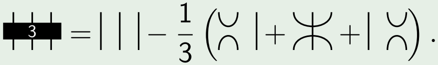



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# Rank Two: Symmetry Property

**Theorem**

*If $\sigma$ is any permutation on three letters, then*

$$\chi_{\sigma(\alpha,\beta,\gamma)}(\sigma(y,x,z)) = \chi_{\alpha,\beta,\gamma}(y,x,z).$$

**Proof.**

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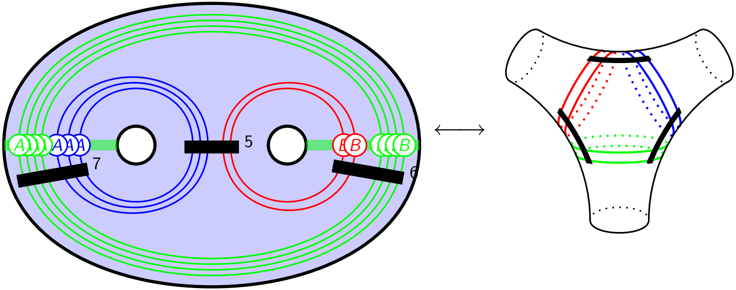

# Rank Two: Recursion Property

Define $\hat{\chi}_{\alpha,\beta,\gamma} \equiv a!b!c!\chi_{\alpha,\beta,\gamma}$ and $\delta \equiv \alpha + \beta + \gamma$ (called the *rank*).
Define $\overline{\alpha} \equiv \alpha + 1$ and $\underline{\alpha} \equiv \alpha - 1$.

## Theorem (Rank Two Recursion)

$$\hat{\chi}_{\alpha,\beta,\gamma} = x \cdot ac\hat{\chi}_{\alpha,\underline{\beta},\gamma} - \gamma^2\hat{\chi}_{\overline{\alpha},\underline{\beta},\underline{\gamma}} - \alpha^2\hat{\chi}_{\underline{\alpha},\underline{\beta},\overline{\gamma}} - \delta^2(\beta-2)^2\hat{\chi}_{\alpha,\underline{\underline{\beta}},\gamma}.$$

**Proof Idea.** Use the fusion identity to join the terms $x = \mathrm{tr}(A)$ and $\chi_{\alpha,\underline{\beta},\gamma}$, and the bubble identity to reduce the result back to the standard form of central functions.

**Notes.**

- As a corollary, the leading term of $\chi_{\alpha,\beta,\gamma}$ is $x^\beta y^\alpha z^\gamma$.
- Together with the symmetry property, this provides an efficient technique for computing central functions.

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**Proof Idea.** Use the fusion identity to join the terms $x = \mathrm{tr}(A)$ and $\chi_{\alpha,\underline{\beta},\gamma}$, and the bubble identity to reduce the result back to the standard form of central functions.

**Notes.**

- As a corollary, the leading term of $\chi_{\alpha,\beta,\gamma}$ is $x^\beta y^\alpha z^\gamma$.
- Together with the symmetry property, this provides an efficient technique for computing central functions.

# Rank Two: Recursion Property

Define $\hat{\chi}_{\alpha,\beta,\gamma} \equiv a!b!c!\chi_{\alpha,\beta,\gamma}$ and $\delta \equiv \alpha+\beta+\gamma$ (called the *rank*).
Define $\overline{\alpha} \equiv \alpha+1$ and $\underline{\alpha} \equiv \alpha-1$.

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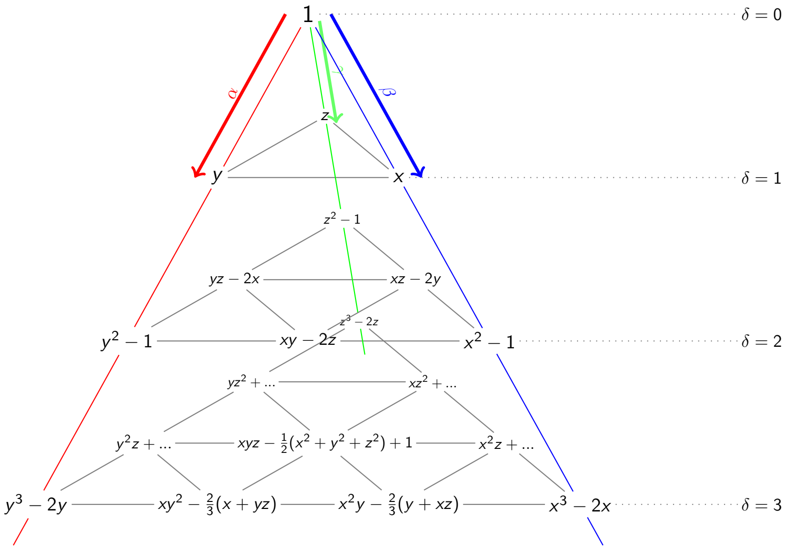

# Rank Two: Ring Structure

## Theorem (Rank Two Product)

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## Idea of Proof.

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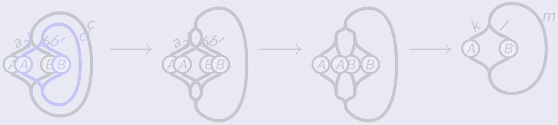

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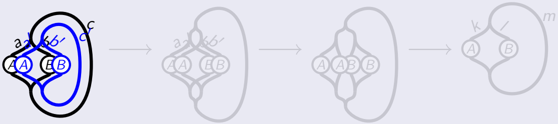

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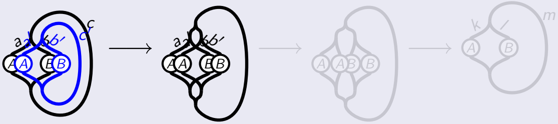

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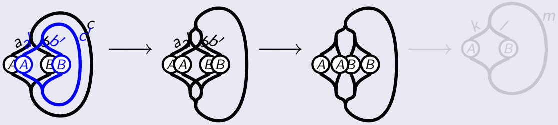

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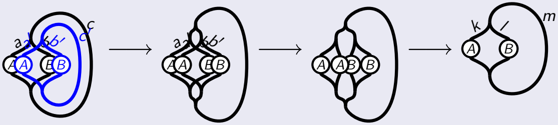

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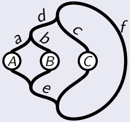
**Definition**

The rank three central functions for $\mathbb{C}[\mathfrak{X}_3]$ are:

$$\chi^{a,b,c}_{d,e,f}(A,B,C) =$$ 

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where the triples $\{a,b,d\}$, $\{a,b,e\}$, $\{c,d,f\}$, and $\{c,e,f\}$ are all admissible.

**Remark.** There are *many* choices of diagram for rank three central functions, and the polynomials obtained will be very different depending on how they are drawn.

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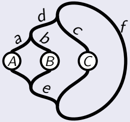


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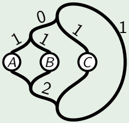
# Rank Three: Sample Computation

**Example**

Compute $\chi^{1,1,1}_{0,2,1} =$ 

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# Rank Three: Partial Table

The interesting cases are when $a, b, c \neq 0$; otherwise, the functions reduce to rank two central functions. Also, either $d \neq 0$ or $e \neq 0$; otherwise, the diagram is disconnected.

Let $x = \mathrm{tr}(A)$, $y = \mathrm{tr}(B)$, $z = \mathrm{tr}(C)$, $X = \mathrm{tr}(B\bar{C})$, $Y = \mathrm{tr}(A\bar{C})$, and $Z = \mathrm{tr}(A\bar{B})$.

**Case** $a = b = c = 1$:

- $\chi^{1,1,1}_{0,2,1} = \frac{1}{2}zZ - X$
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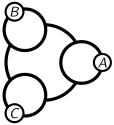
1. The Central Function Basis
   - Algebraic Approach
   - Diagrammatic Approach
2. Trace Diagrams and Representation Theory
   - Representations and Tensor Algebra
   - $SL(2,\mathbb{C})$ Trivalent Diagrams
3. Computation of Central Functions
   - Rank One
   - Rank Two
   - Rank Three
4. Questions for Exploration
   - Computing $SL(2,\mathbb{C})$ Central Functions
   - Generalizations

# Central Functions for Rank Three and Beyond

There are a number of challenges to overcome before the general theory of central functions can be full developed. For a systematic approach to work, central functions should be defined in a standard, symmetric way for all ranks. There are many ways to do this, for example:

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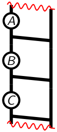


The advantage in either case is that the functions are built up from several identical components. This should provide a standard technique for developing recursion and product formulas, as well as computational algorithms.

# Central Functions and Surface Structure

**Question.** How can trace diagrams take into account the structure of the *surface* as well as its fundamental group?

**Partial Answer.** Use the Poisson structure!

**Definition**

The Goldman bracket $\{f, g\}$ of two loops on a surface is the sum over all essential intersections of the following:




- This bracket satisfies the Jacobi and Leibniz identities, and so gives the ring a Poisson structure.
- The bracket is simply the application of the binor identity  to the essential crossings in a diagram.

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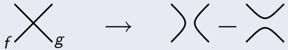


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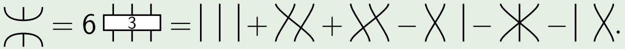
# Generalizations: Higher Dimensions

To generalize to $n \times n$ matrices, generalize :

## Example

Trace diagrams for $3 \times 3$ matrices are trivalent graphs, with antisymmetrizer



The local maxima and minima are defined via

 $: \mathbb{C} \to V \otimes V \otimes V;$

$$1 \mapsto e_1 \otimes e_2 \otimes e_3 + \cdots - e_1 \otimes e_3 \otimes e_2.$$

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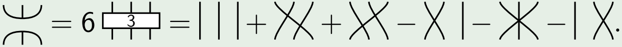


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# Central Functions and Lie Algebras

**Question.** Can the theory of central functions be developed using Lie algebras rather than Lie groups?

**Answer.** Yes... I think! One interesting fact is that there is a nice diagram for transforming a matrix $X \in \mathrm{SL}(2,\mathbb{C})$ into a matrix $x \in sl(2,\mathbb{C})$, since the result necessarily has trace 0. This is the mapping

$X \mapsto X\frac{1}{2}\mathrm{tr}(X)I$. Such diagrams are the primary "building blocks" of one type of central function:

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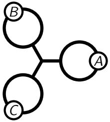

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**Question.** How does this all relate to knot theory and quantum groups?

**Answer.**

- Central functions are very closely related to the theory obtained from "quantizing" crossings. The correspondence is exact in rank one.
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- Sean Lawton
- Bill Goldman
- Charlie Frohman


- Predrag Cvitanovic, *Group Theory*, http://chaosbook.org/GroupTheory/
- Carter/Flath/Saito, *The Classical and Quantum 6j-Symbols*