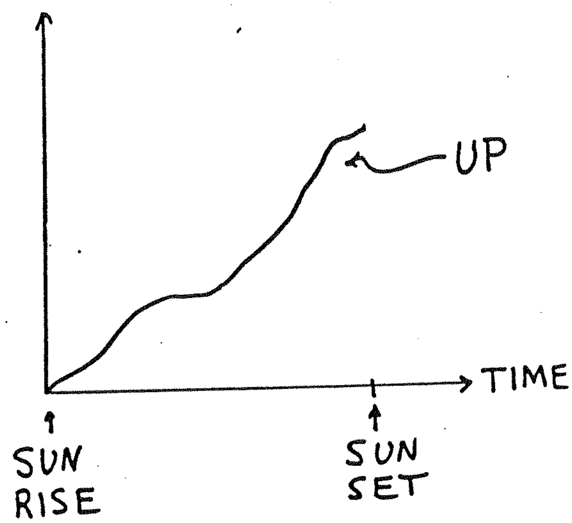
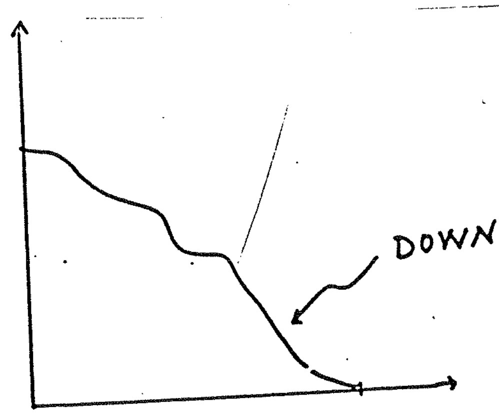
Bolzano's Intermediate Value Theorem

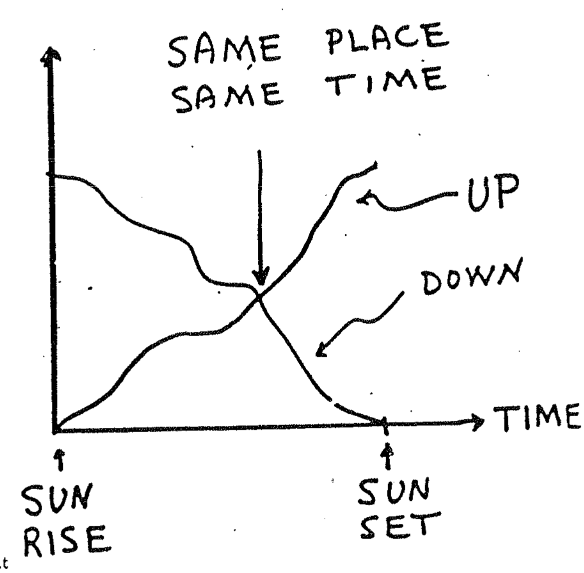
When introducing the intermediate value theorem I begin with the following story that ends with a question (taking care to pose it at the end of a class, so the students have an evening to think about it):

> One day a monk leaves at sunrise to climb up a mountain. He walks at a leisurely pace, sometimes stopping to enjoy the view, even retracing his path to look again at a pretty flower. He arrives at the summit at sundown, spends the night meditating, and starts home down the same path the next day at sunrise, arriving home at sunset. The question is this: Was there a time of day when he was exactly at the same point on the trail on the two days?





The next class begins by bringing out the hidden assumptions of this ill posed problem (doing this is one of the strengths of the problem). I do this by getting the students to talk about the problem, and, most often, why they cannot solve it. The monk travels along the same path on both days and his position is determined by the distance from the bottom of the path. Position is, of course, a continuous function of time. If we plot the path up the mountain in a time-distance coordinate system, then the curve goes from (sunrise, bottom) to (sunset, top). Flat regions on the graph are rest times, and dips arise from, say, retracing his steps to look at a flower. The path down the mountain is a curve from the point (sunrise, top) to (sunset, bottom). When the two paths are plotted on the same axes, it is obvious that the curves intersect—this is a point where the monk is at the same point at the same time on the two days.



There is an insightful solution to the problem that is equivalent to this graphical one, but there is no need to draw any picture. One monk and two days does not make the solution as transparent as it could be so we use Polyá's technique of looking at a similar (and in this case, equivalent) problem. Suppose there are two monks and they both start at sunrise, one at the bottom of the mountain, the other at the top. Every student will see that they must meet somewhere along the path—at the

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Then, I point out that the theorem we used here, which we call the Intermediate Value Theorem, is due to Bernhard Bolzano (1781–1848), who was, in fact, a monk. His mother was a pious woman, his father an Italian immigrant who earned a modest living as an art dealer. His father was widely read and felt responsible for his fellow men. This was not just theory; he took an active part in founding an orphanage in Prague. His son, Bernhard, studied philosophy, physics, and mathematics at the University of Prague in his native city. It was this grounding in philosophy and logic that convinced him of the necessity of formulating clear concepts and of using sound reason to deduce theorems from irreducible first principles. His interest in mathematics was stimulated by B. Kästner's *Anfangsgründe der Mathematik*, a book where the author took care to prove statements which were commonly regarded as evident in order to make clear the assumptions on which they depended.[2]

After graduating in 1800, Bolzano entered the theological faculty at the University of Prague and was ordained a Roman Catholic priest in 1804. In 1805, Emperor Franz I of Austria, of which Bohemia was then a part, established a chair of philosophy in each university. His reasons were mainly political, as he feared the spread of the ideas which had fomented the French Revolution and which were widespread in Bohemia. In 1805 Bolzano was appointed to the new chair of Philosophy of Religion at the University of Prague. His unorthodox religious and political ideas made him quite unsuitable for this position. However his lectures and sermons were exceptionally popular among the students in that he advocated human rights and utopian socialism. These views, as well as his unorthodox religious views, led to his dismissal on 24 December 1819. He was forbidden to publish and was put under police supervision, but he refused to recant. The remainder of his life was spent working on philosophy and mathematics

1781–1848

BERNARD BOLZANO

Engraving by Schulz after a drawing by Kriehuber, published in *Starý Světozor*, 1908. (Picture by courtesy of the Museum of Czech Literature.)

while living with friends.

In Prague he was isolated from the center of the mathematical world in Paris. The fact that he held a university post for only a few years also contributed to the fact that his mathematical ideas received little recognition until Herman Hankel and Otto Stolz called attention to them in 1871 and 1881 respectively.[3] Then they rapidly became well known.

But we are way ahead of the story. The work that interests us is Bolzano's now famous "Rein analytischer Beweis" of 1817, which has the full title "Purely analytic proof of the theorem that

1 See James C. Frauenthal and Thomas L. Saaty, "Foresight—Insight—Hindsight," pp. 1–22 (especially pp. 3–4) in *Discrete and System Models* edited by William F. Lucas, Fred S. Roberts, and Robert M. Thrall, Springer, 1976. This is volume 3 in the series *Modules in Applied Mathematics*.

2 For additional information about Bolzano, see B. Van Rootselaar, "Bolzano, Bernard," *Dictionary of Scientific Biography*, vol. 2, pp. 273–279.

3 Herman Hankel (1839–1873), "Grenze," *Allg. Encl. Wiss. Kunst* (1871), sect. 1, part 90, pp. 185-211, Leipzig. Otto Stolz (1842–1905) "B. Bolzano's Bedeutung in der Geschichte der Infinitesimalrechnung," *Mathematische Annalen*, 18 (1881), pp. 255–279.

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between any two values which give results of opposite sign there lies at least one real root of the equation."[4] This is the theorem which we now call the intermediate value theorem. In this paper he also gives the definition of continuity that we still use today.

Bolzano makes the claim that his theorem "clearly rests on the more general truth that, if two continuous functions of x, fx and ϕx, have the property that for $x = \alpha$, $f\alpha < \phi\alpha$, and for $x = \beta$, $f\beta > \phi\beta$, there must always be some value of x lying between α and β for which $fx = \phi x$."[5] Although Bolzano doesn't draw a picture we should draw one for our students, because then the solution to the monk's problem becomes transparent.

Bolzano made a distinction in this paper which we should heed as teachers. He is not interested in giving a mere 'confirmation' of the theorem, but wished to give a 'justification' of it. He points out that the theorem is perfectly obvious and does not need confirmation. He provides a justification or proof of the theorem because he is interested in the foundations of analysis, a pursuit in which he was far ahead of his contemporaries. If we don't want to be far ahead of our students then we should dispense with a justification of the intermediate value theorem and concentrate on it's confirmation. This is a case where history tells us what not to prove in the classroom.

The same mathematical results are in Cauchy's *Cours d'Analyse* of 1821. This coincidence has prompted Grattan-Guinness, in his doctoral dissertation, to look for an explanation of these similarities.[6] He concluded that Cauchy stole this definition from Bolzano, citing similarities in the work, the fact that the journal which carried Bolzano's paper began to appear in the Paris libraries with the very issue in which the "Rein Beweis" appeared, the fact that Cauchy read German, was careless in citing his sources, and was a rather nasty man. This charge is now generally regarded as unfounded [Freudenthal, Sinaceur, as well as much work done recently on Bolzano that does not specifically deal with this *ex post facto* priority dispute.] Grabiner concludes that while Grattan-Guinness gave the wrong answer, he did ask the right question.[7]

The Intermediate Value Theorem is a theoretical tool in our calculus classes today (I am still working on how it attained this role in the nineteenth-centruy). There are very few applications. The type of problems that is most prevalent in our texts, that of finding the point c where the function assumes the intermediate value, is a bogus problem, and should be banned. If we knew how to find c we would not need to use the IVT. There are only a few problems that I know of that make essential use of the IVT:

(1) From my classroom in Bowling Green, I ask if there is a direction that I can point such that the temperature at the boundary of the State of Ohio is the same in that direction and in the opposite direction? I hold my arms out at my sides and point in opposite directions, and then swing around as I ask the question. Define a function as the difference in temperatures at the state boundary between where my right hand points and where my left points. Fix a direction. If the function is zero there, we are done. If not, turn 180°. For this direction the function will have the opposite value. By the IVT, the function must be zero for some intermediate direction. I don't know if this argument will work in your state, but if it fails, it will be instructive to see why.

(2) Must it happen at some time in your life that your height in feet equals your age in years?

(3) Stretch a rubber band. Is there some point that does not move?

These are the only interesting problems that I know which use the Intermediate Value Theorem. Part of their charm is that each of them has hidden assumptions and conditions which need to be brought out. If you have other such problems, I would be happy to hear of them. I would also like to know about the history of these problems, but I haven't a clue.

[4] "Rein analytischer Beweis des Lehrsatzes dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reele Wurzel der Gleighung liege," *Abh. Gesell. Wiss. Prague*, (3), vol. 5, (1814–1817), pp. 1–60. English translation by S. B. Russ in *Historia Mathematica*, vol. 7(1980), 156-185.

[5] Bolzano, op. cit., p. 166]

[6] "Bolzano, Cauchy and the 'new analysis' of the early nineteenth century," *Archiv for the History of Exact Sciences*, vol. 6 (1970), pp. 372–400. *The Development of the Foundation of Mathematical Analysis from Euler to Riemann*, MIT press, 1970.

[7] Judith V. Grabiner, 'Cauchy and Bolzano. Tradition and transformation in the history of mathematics," pp. 105–124 in *Transformation and Tradition in the Sciences* (1984), edited by Everett Mendelsohn, Cambridge University Press.

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