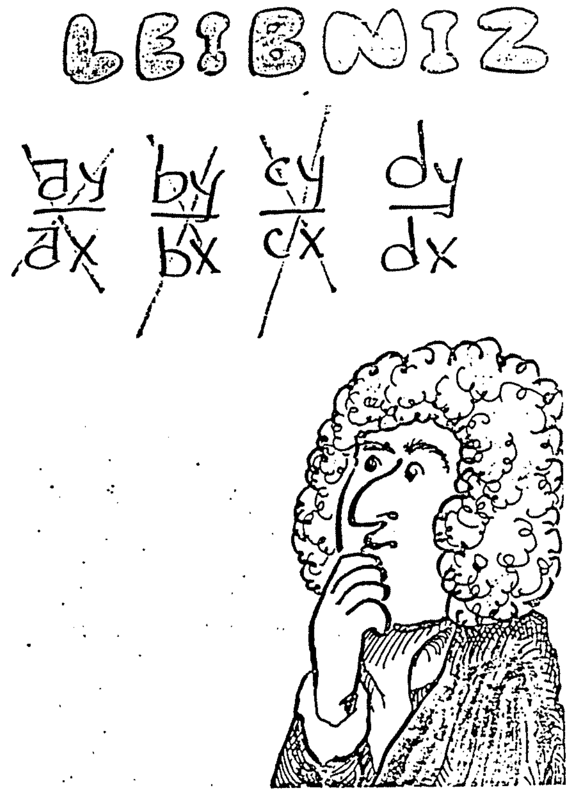
The Product Rule



One thing we should not neglect to tell our students, for it is especially encouraging to the beginner, is that Leibniz had considerable difficulty discovering the correct form of the product rule. It is exciting to follow his struggle in his manuscripts during 1675–1676 when he was in the process of inventing the calculus. Indeed, these are some of the most precious documents in the whole history of mathematics. Among other things, they provide a wonderful example of how mathematics is done.

In a manuscript dated 11 November 1675, Leibniz introduced the differential notation dx. He thought of a variable as taking on a sequence of values and he was considering *differences* of these.[1] This is why he chose the letter d — it stands for 'difference,' not for 'differential' or 'derivative' as we are inclined to believe today.[2] My colleague Vic Norton conceives of Leibniz's choice of notation in a more humorous way (see the figure).

In this same manuscript of 11 November, 1675, Leibniz wrote "Let us now examine whether $dx\,dy$ is the same thing as $d\overline{xy}$, and whether dx/dy is the same thing as $d\frac{x}{y}$." Here he has used the overbar where we would use parentheses. In order to test this conjecture he considered an example: He took $x = cz + d$ and $y = z^2 + bz$ and then correctly computed $dx\,dy$. Then in the rush of discovery he added, "But you get the same thing if you work out $d\overline{xy}$ in a straightforward manner." But he neglected to do it! Consequently, we have the makings of a good problem to give our students—continue the example and draw the conclusion that Leibniz should have drawn.

Later in the same manuscript, after noting the absurdity of $\int \overline{d\nu\, d\psi} = \int d\nu \int d\psi$, he writes[3]

> Hence it appears that it is incorrect to say that $d\nu\, d\psi$ is the same thing as $d\nu\psi$, or that $\frac{d\nu}{d\psi} = d\frac{\nu}{\psi}$; although just above I stated that this was the case, and it appeared to be proved. This is a difficult point. But now I see how this is to be settled.

It is not clear what he meant by "appeared to be proved" but he settled the difficulty by counterexample, taking $\nu = \psi = x$, augmenting x by dx, and then computing:

$$d(x^2) = (x + dx)^2 - x^2 = 2xdx$$

without even a mention of what happened to the $(dx)^2$. Next he wrote

$$dx\,dx = (x + dx - x)(x + dx - x) = (dx)^2.$$

[1] Henk J. M. Bos, "The fundamental concepts of the Leibnizian calculus," *Studia Leibnitiana*, Sonderheft 14 (1986), 103-118; reprinted in his *Lectures in the History of Mathematics* (1993), AMS, pp. 83-99.

[2] The concept of derivative came much later. See Judith V. Grabiner, "The changing concept of change: The derivative from Fermat to Weierstrass," *Mathematics Magazine*, 56 (1983), pp. 195–203; reprinted in Frank Swetz, *From Five Fingers to Infinity* (1994), pp. 607-619.

[3] Leibniz had invented the integral sign thirteen days previously, on October 29, 1675.

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In this paragraph we have changed the notation, while previously Leibniz's notation has been carefully preserved.

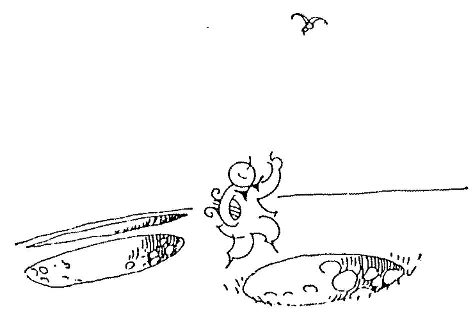
Ten days later, on 21 November 1675, Leibniz has the product rule, but stated in the form

$$\overline{dx}\,y = \overline{dxy} - x\,\overline{dy}.$$

He notes "this is a really noteworthy theorem and a general one for all curves." Then he cryptically adds "But nothing new can be deduced from it, because we had already obtained it."

By way of encouragement and motivation to the student we should point out that it took Leibniz ten days to figure out the product rule. But then he had to discover it. They only have to learn to use it. But then they had better do that in ten days—or risk flunking the next exam.

I also point out that there is nothing wrong with making mistakes. This example shows that one of the greatest minds of all times made mistakes. What is wrong is not to continue to think about what you have done until you are sure that everything is OK. The following 'grook' says this in a more positive light.[4]



THE ROAD TO WISDOM

The road to wisdom? —Well, it's plain
and simple to express:
Err
and err
and err again
but less
and less
and less.

It is not until Leibniz's manuscripts of July 1677 that we find what might reasonably be called a proof of the product rule, but we shall not quote it here[5] because the following proof, given in a letter which Leibniz wrote Wallis on March 30, 1699, while essentially the same, is somewhat clearer:

> It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero, but which are rejected as often as they occur with quantities incomparably greater. Thus if we have $x + dx$, then dx is rejected. But it is different

[4] Piet Hein, *Grooks*, MIT press, 1966. Piet Hein (b. 1905) is a Danish engineer, poet, and intellectual jack-of-all-trades. As a friend of many mathematicians, he has applied his skills to both architecture and games. He invented the "super-ellipse" $|\frac{x}{a}|^p + |\frac{y}{b}|^p = 1$ as the shape of a traffic circle for a rectangular "square" in Stockholm City Center. In 1942 Hein invented the game of Hex (the American mathematician and Nobel laureate in Economics John Nash proved that the first player always wins). He also invented the SOMA Cubes which perplexed thousands in the 1960s (the name is a registered trademark of Parker Brothers). His "Grooks" are delightful, short, aphoristic poems, each accompanied by one of his drawings. Originally they were a kind of underground language—beyond the German comprehension, and way beyond their sensibilities—used by the Danish Resistance to the Nazi's in World War II. More recent Grooks in English apply his wit and wisdom to the human condition. Several are ideal for use in the classroom. Hein is truly a great Dane. For a picture of Hein see p. 328 of Anatole Beck, Michael N. Bleicher, and Donald W. Crowe, *Excursions into Mathematics*, New York: Worth, 1969.

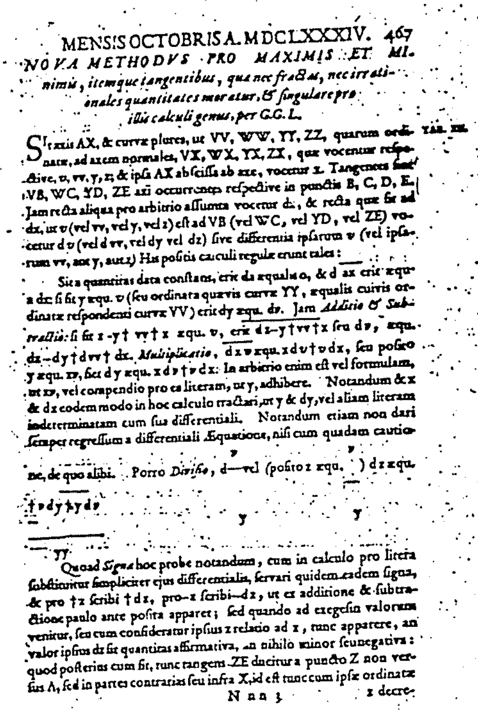
[5] All of the above quotations are taken from "The manuscripts of Leibniz on his discovery of the differential calculus," *The Monist*, **26**(1916), 577–629, **27**(1917), 238–294 and 411–454; esp. pp. 279–281, 286 and 439. This paper is a translation, with highly unreliable commentary by J. M. Child, of papers by C. I. (or K. J.) Gerhardt, who discovered the papers in the Royal Library of Hanover in the mid-nineteenth century. They are also in J. M. Child's *The Early Mathematical Manuscripts of Leibniz. Translated from the Latin Texts Published by Carl Immanuel Gerhardt with Critical and Historical Notes.* Chicago, London: Open Court, 1920, iv + 238 pp. See pages 100–102, 107 and 143 for the passages quoted.

if we seek the difference between $x+dx$ and x, for then the finite quantities disappear. Similarly we cannot have $x\,dx$ and $dx\,dy$ standing together. Hence if we are to differentiate xy we write:

$$(x+dx)(y+dy) - xy = x\,dy + y\,dx + dx\,dy$$

But here $dx\,dy$ is to be rejected as incomparably less than $x\,dy + y\,dx$. Thus in any particular case the error is less than any finite quantity.[6]

The product rule first appeared in print in 1684 in Leibniz's first paper on the differential calculus: "A new method for maxima and minima as well as tangents which is neither impeded by fractional nor irrational quantities, and a remarkable type of calculus for them." See if you can locate the statement of the product rule on the first page of this famous paper of Leibniz: [7]



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NOVA METHODUS PRO MAXIMIS ET MINIMIS, itemque tangentibus, quæ nec fractas, nec irrationales quantitates moratur, & singulare pro illis calculi genus, per G.G.L.

Sit axis AX, & curvæ plures, ut VV, WW, YY, ZZ, quarum ordinatæ, ad axem normales, VX, WX, YX, ZX, quæ vocentur respective, v, w, y, z; & ipsa AX abscissa ab axe, vocetur x. Tangentes sint VB, WC, YD, ZE axi occurrentes respective in punctis B, C, D, E. Jam recta aliqua pro arbitrio assumta vocetur dx, & recta quæ sit ad dx, ut v (vel w, vel y, vel z) est ad VB (vel WC, vel YD, vel ZE) vocetur dv (vel dw, vel dy vel dz) sive differentia ipsarum v (vel ipsarum w, aut y, aut z) His positis calculi regulæ erunt tales:

Sit a quantitas data constans, erit da æqualis 0, & d ax erit æqu. a dx: si sit y æqu. v (seu ordinata quævis curvæ YY, æqualis cuivis ordinatæ respondenti curvæ VV) erit dy æqu. dv. Jam *Additio & Subtractio*: si sit z -y + w + x æqu. v, erit dz-y+w+x seu dv, æqu. dz-dy+dw+dx. *Multiplicatio*, dxv æqu. xdv+vdx, seu posito y æqu. xv, fiet dy æqu. xdv+vdx. In arbitrio enim est vel formulam, ut xv, vel compendio pro ea literam, ut y, adhibere. Notandum & x & dx eodem modo in hoc calculo tractari, ut y & dy, vel aliam literam indeterminatam cum sua differentiali. Notandum etiam non dari semper regressum a differentiali Æquatione, nisi cum quadam cautione, de quo alibi. Porro *Divisio*, d v/y vel (posito z æqu. v/y) dz æqu. +vdy∓ydv / yy

Quoad *Signa* hoc probe notandum, cum in calculo pro litera substituitur simpliciter ejus differentialis, servari quidem eadem signa, & pro +z scribi +dz, pro -z scribi -dz, ut ex additione & subtractione paulo ante posita apparet; sed quando ad exegesin valorum venitur, seu cum consideratur ipsius z relatio ad x, tunc apparere, an valor ipsius dz sit quantitas affirmativa, an nihilo minor seu negativa: quod posterius cum fit, tunc tangens ZE ducitur a puncto Z non versus A, sed in partes contrarias seu infra X, id est tunc cum ipsæ ordinatæ z decre-

N n n 3

It next occurred in Newton's *Philosophiae naturalis principia mathematica*, book 2, lemma 2, although Newton had known the result since 1665.[8] Newton's proof of the result was harshly criticized by George Berkeley in his *Analyst* of 1734. "The *Analyst* is more than a Christian apologetic; it is a work on mathematics for mathematicians,"[9] and profoundly influenced the development of the foundations of the calculus. Here is what Berkeley says about the product rule:

> I proceed to consider the principles of this new analysis ... wherein if it shall appear that your capital points, upon which the rest are supposed to depend, include error and false reasoning; it will then follow that you, who are at a loss to conduct your selves, cannot with any decency set up for guides to other men. The main point in the method of fluxions is to obtain the fluxion or momentum of the rectangle or product of two indeterminate quantities. ... Now, this fundamental point one would think should be very clearly made out, ... But let the reader judge. This is given as demonstration [by Newton]. Suppose the product or rectangle AB increased by continual motion: and that the momentaneous increments of the sides A and B are a and b. When the sides A and B were deficient, or lesser by one half of their moments, the rectangle was $\overline{A-\frac{1}{2}a} \times \overline{B-\frac{1}{2}b}$ i.e., $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$. And as soon as the sides A and B are increased by the other two halves of their moments, the rectangle becomes $\overline{A+\frac{1}{2}a} \times \overline{B+\frac{1}{2}b}$ or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. From the latter rectangle subduct the former, and the remaining difference will be $aB + bA$. Therefore the increment of the rectangle generated by the entire increments a an b is $aB + bA$. *Q.E.D.* But it is plain that the direct and true method to obtain the moment or increment of the rectangle AB, is to take the sides as increased by their whole increments, and so multiply them together, $A+a$ by $B+b$, the product whereof

[6] *Leibnitzens Mathematische Schriften*, IV, 63, edited by Gerhardt. Translation from D. E. Smith, *History of Mathematics*, vol. 2, p. 696–697.

[7] For an English translation of this paper, see D. J. Struik, *A Source Book in Mathematics, 1200–1800*, pp. 271–280.

[8] *The Mathematical Papers of Isaac Newton. Volume 1. 1664-1666*, edited by D. T. Whiteside, Cambridge University Press, 1967, pp. 344, 383, and 402.

[9] So say A. A. Luce and T. E. Jessop on p. 58 of the editor's introduction to *The Works of George Berkeley, Bishop of Cloyne*, London: Thomas Nelson and Sons, 1951, vol. 4.

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$AB + aB + bA + ab$ is the augmented rectangle; whence, if we subduct AB the remainder $aB + bA + ab$ will be the true increment of the rectangle, exceeding that which was obtained by the former illegitimate and indirect method by the quantity ab. And this holds universally be the quantities a and b what they will, big or little, finite or infinitesimal, increments, moments, or velocities. Nor will it avail to say that ab is a quantity exceedingly small: since we are told that *in rebus mathematicis errores quam minimi non sunt contemnendi.* [The most minute errors are not in mathematical matters to be scorned.][10]

Berkeley's criticism here is right on the mark. Mathematicians were unable to give a better proof of the product rule until Cauchy introduced the definition of the derivative using limits in his *Cours d'Analyse* of 1821.

Exercises

1. "A function $u(x)$ being given, it is required to determine a formula giving all the functions $v(x)$ for which the derivative of the product u and v is equal to the product of their derivatives."[11]
2. Verify this alternate proof of the product rule: First get $\frac{d}{dx}[f^2(x)]$ from the definition (which is an interesting exercise anyway) and then use the identity $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$ to finish the proof.[12]
3. Maria Agnesi in provided an easy approach to the quotient rule in her *Instituzione Analitiche* of 1748: If $h = f/g$, then $hg = f$. Now apply the product rule, substitute f/g for h and then solve for h'.


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Typis CHRISTOPHORI GÜNTHERI.
Anno MDCLXXXIV.

[10] Berkeley's footnote is *"Introd. ad Quadraturam Curvarum."* This refers to Newton's "Tractatus de quadratura curvarum," the second appendix to his*Optics* (1704). The line Berkeley quotes in Latin is from page 167, but he permuted the word order. See *The Mathematical Papers of Isaac Newton*, vol. VIII, pp. 124–5. The long passage quoted above is from Berkeley's *The Analsyt* (1737) which has been reprinted in Luce and Jessop, op. cit., vol. 4, pp. 53–102; see §9, pp. 69–70.

[11] This problem is from the 0$^{\text{th}}$ Putnam exam, which was held May 19 and 20, 1933. See David C. Arney, "Army beats Harvard in football and mathematics," *Math Horizons*, September 1994, pp. 14–17. Of course Army won; the score was 112 to 98.

[12] Russell Euler, "A note on differentiation," *The College Mathematics Journal*, 17(1986), 166–167.